

*Embeddings of expanders in Banach spaces.*

In problem 3.1 of the list of open problems of 2003 (see “Open problems on embeddings of finite metric spaces” at [http://kam.mff.cuni.cz/\(tilde\)matousek/dg.html](http://kam.mff.cuni.cz/(tilde)matousek/dg.html)), Linal asked the following question :

Let  $X$  be a infinite-dimensional Banach space. Suppose there is a family of constant degree expanders  $G_k$  on  $n_k$  vertices,  $n_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ , that embed in  $X$  with distorsion  $o(\log n_k)$ . Does it imply that  $X$  has no cotype?

I proved (see [http://www.institut.math.jussieu.fr/\(tilde\)vlaflorg/edimbourg.pdf](http://www.institut.math.jussieu.fr/(tilde)vlaflorg/edimbourg.pdf) and [http://www.institut.math.jussieu.fr/\(tilde\)vlaflorg/Trenforce.pdf](http://www.institut.math.jussieu.fr/(tilde)vlaflorg/Trenforce.pdf)) that if  $X$  has type  $> 1$ , there is a family of expanders which do not embed in  $X$  with distorsion  $o(\log n_k)$  (and a fortiori which do not embed uniformly in  $X$ ). Unfortunately

1. the proof uses the fact that some average operators on the subspace  $\ell_0^2(G_{n_k}, X)$  of functions with average 0 in  $\ell^2(G_{n_k}, X)$  have norm  $< 1$  and this is false when  $X$  has type 1
2. the expanders are built from lattices in  $SL_3(F)$ ,  $F$  a local non-archimedean local field, and the result comes from a strong form of property (T), and therefore the proof does not extend to Ramanujan graphs.

Remark about 1 : it is known that a type  $> 1$  Banach space has a finite cotype (Maurey-Pisier), but the converse is false, because  $\ell^1$  has type 1 but cotype 2. Assaf Naor asked the following question (which would imply that a result of non uniform embeddability in type  $> 1$  Banach spaces implies the same result for finite cotype Banach spaces) :

does every Banach space with finite cotype embed uniformly in a Banach space with type  $> 1$ ?

Remark about 2 : if  $G_k$  a family of constant degree expanders on  $n_k$  vertices,  $n_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ , and  $\hat{A}_k$  is the normalized adjacency matrix on the subspace  $\ell_0^2(G_{n_k}, X)$  of functions with average 0 in  $\ell^2(G_{n_k}, X)$ , the following estimate would imply that  $G_k$  do not embed in  $X$  with distorsion  $o(\log n_k)$  :

$$\lambda = \sup_k \left\| \frac{1 + \hat{A}_k}{2} \right\|_{\mathcal{L}(\ell_0^2(G_{n_k}, X))} < 1 \tag{1}$$

because for  $f \in \ell_0^2(G_{n_k}, X)$ , we would have

$$\|f\|_{\ell_0^2(G_{n_k}, X)} \leq \frac{1}{2(1-\lambda)} \|f - \hat{A}_k f\|_{\ell_0^2(G_{n_k}, X)} \leq C \sum_{x,y \text{ neighbors}} \|f(x) - f(y)\|^2.$$

On the other hand  $\hat{A}_k$  satisfies

$$\left\| \frac{1 + \hat{A}_k}{2} \right\|_{\mathcal{L}(\ell_0^1(G_{n_k}, \mathbb{R}))} = 1, \left\| \frac{1 + \hat{A}_k}{2} \right\|_{\mathcal{L}(\ell_0^\infty(G_{n_k}, \mathbb{R}))} = 1, \text{ and } \sup_k \left\| \frac{1 + \hat{A}_k}{2} \right\|_{\mathcal{L}(\ell_0^2(G_{n_k}, \mathbb{R}))} < 1.$$

Is this enough to imply (1) if  $X$  is uniformly convex?

More generally if  $X$  is a uniformly convex Banach space, and  $a < 1$ , does there exist  $b < 1$  such that for any finite set  $I$  and any square matrix  $T$  indexed by  $I$  satisfying

$$\|T\|_{\mathcal{L}(\ell^1(I, \mathbb{R}))} \leq 1, \|T\|_{\mathcal{L}(\ell^\infty(I, \mathbb{R}))} \leq 1, \|T\|_{\mathcal{L}(\ell^2(I, \mathbb{R}))} \leq a,$$

then  $\|T\|_{\mathcal{L}(\ell^2(I, X))} \leq b$ ? We can also ask  $b \leq Ca^\alpha$  for some  $\alpha > 0$ .

Discussion : Naor and Schechtman point out a counter-example by Pisier of uniformly convex but not UMD Banach spaces, and an article of Figiel. The answer to the last general question is very probably no.

To extend my result quoted above to expanders constructed from lattices in  $SL_3(\mathbb{R})$ , I would need a particular case of the last question. Let  $S^2$  the sphere in the euclidian  $\mathbb{R}^3$ , and for any  $\epsilon \in ]-1, 1[$ , define the operator  $T_\epsilon : L^2(S^2, \mathbb{R}) \rightarrow L^2(S^2, \mathbb{R})$ , by

$$T_\epsilon(f)(x) = \frac{1}{2} \left( \int_{y \text{ on the circle } \langle y, x \rangle = 0} f(y) dy - \int_{y \text{ on the circle } \langle y, x \rangle = \epsilon} f(y) dy \right)$$

where both integrals are averages for the uniform measure on the circles.

Is it easy to see that

$$\|T_\epsilon\|_{\mathcal{L}(L^1(S^2, \mathbb{R}))} \leq 1 \text{ and } \|T_\epsilon\|_{\mathcal{L}(L^\infty(S^2, \mathbb{R}))} \leq 1$$

and it is proven in <http://www.institut.math.jussieu.fr/~vlafforg/Trenforce.pdf> that for some constant  $C$ ,

$$\|T_\epsilon\|_{\mathcal{L}(L^2(S^2, \mathbb{R}))} \leq C\sqrt{|\epsilon|}.$$

Question : if  $X$  is a uniformly convex Banach space, does there exist  $C$  and  $\alpha > 0$  such that for any  $\epsilon \in ]-1, 1[$ ,

$$\|T_\epsilon\|_{\mathcal{L}(L^2(S^2, X))} \leq C|\epsilon|^\alpha?$$