How to make \$ 7000

1 Conditions

I will award the prizes below as long as I am not too senile to understand the proofs I receive.

Please note that you do not need to be the author of the solution in order to get the money, it is just fine if you buy it from somebody else for half the amount I offer.

2 Make \$ 5000 by proving the Bernoulli conjecture

This conjecture in explained in great detail in my book “the generic chaining” with Springer. A negative solution will get you \$ 1000.

After the book was published, Rafal Latala solved the problem at the bottom of page 144. His paper, “Boundedness of Bernoulli processes over thin sets”, is in ECP 13, 2008, pp. 175-188.

Please also note that (as discovered by Jian Ding and James Lee) the example given in Section 4.4 page 144 is nonsense. The chopping maps, used for $c = 2^{-k/4}$ do prove that $b(T) \geq k/L$, by combining Theorem 4.2.4 and Proposition 4.3.7.

3 Make \$ 1000 from simple combinatorics

This is problem 5.5.2 of the “the generic chaining” book. You have to have a look at the book to get the connection between this problem and stochastic processes, but the problem can be understood without knowing anything about this book.

Consider a number $\delta \leq 1/2$, an integer N and the product measure P on $\{0,1\}^N$, when on each factor one gives mass δ to 1 and $1-\delta$ to 0. Given an integer q and a set $D \subset \{0,1\}^N$ we define

$$\begin{aligned} D^{(q)} = \{(x_1, \cdots, x_N) \in \{0,1\}^N; \quad & \forall (x_1^1, \cdots, x_N^1), \cdots, (x_1^q, \cdots, x_N^q) \in D, \\ & \exists i \leq N, x_i = 1, x_i^1, \cdots, x_i^q = 0\}. \end{aligned}$$

If one identifies points in $\{0,1\}^N$ with subsets of $\{1,\cdots,N\}$, then $D^{(q)}$ is simply the collection of subsets of $\{1,\cdots,N\}$ that cannot be covered by q elements of D. Also, P is the law of a random set J obtained as follows: for each $i \in \{1,\cdots,N\}$ the probability that $i \in J$ is δ, and these probabilities are independent over i.

Given a subset I of $\{0,1\}^N$, we denote

$$B(I) = \{(x_1,\cdots,x_N) \in \{0,1\}^N; \forall i \in I, x_i = 1\}.$$

This is simply the collection of subsets of $\{1,\cdots,N\}$ that contain I.

Problem. Can one find a number q, independent of N and δ, such that for each set D with $P(D) \geq 1 - 1/q$ we can find a family $\mathcal{G}$ of subsets of $\{1,\cdots,N\}$ such that

$$D^{(q)} \subset \bigcup_{I\in\mathcal{G}} B(I)$$

and

$$\sum_{I\in\mathcal{G}} \delta^{\mathrm{card}I} \leq 1/2?$$

If, rather that the above condition you obtain the weaker condition that

$$\sum_{I\in\mathcal{G}} \delta'^{\mathrm{card}I} \leq 1/2,$$

where δ' depends on δ only, you get the money. (The case $\delta = 1/2$ won't get you anything, $q = 2$ works.)

This problem is discussed in great detail in my paper "Are many small sets explicitly small?", which is available in the section "Preprints" of this site. Problems related to this one (but in some sense less basic) are explained in my paper "Are all sets of positive measure essentially convex?" Operator theory: Advances and applications. Vol. 77, Birkhäuser, 1995, p. 295-310.

4 Add another \$ 1000 for a simple exercise on convolution

This is the convolution problem asked in the paper "A conjecture on convolution operators, and operators from L^1 to a Banach lattice", Israel J. of Math. 68, 1989, p. 82-88.

Consider the group $G = \{-1,1\}^{\mathbb{N}}$, and denote by λ its Haar measure. Denote by μ a "biased coin" measure on G. This is a product measure with identical factors that gives different weights to 1 and -1, or, equivalently, this is the law of an infinite sequence of trials obtained by flipping a biased coin.

Problem. Prove that there is function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{t\to\infty} t\phi(t) = 0$$

$$\forall f \text{ on } G, f \geq 0, \int f d\lambda = 1, \forall t \geq 0,\ \lambda(\{f * \mu \geq t\}) \leq \phi(t).$$

Of course, f is assumed to be measurable. The meaning of this statement is that convolution by a biased coin smooths a function, in the sense that the convolution never looks like a "peak function" $\frac{1}{\lambda(A)}\mathbf{1}_A$ for $\lambda(A)$ small. Of course the function ϕ will depend on μ.