Remarques sur les surfaces de courbure moyenne grande

Remarks on surfaces of large mean curvature

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Abstract

Let M be a complete embedded H-surface in N^3, of bounded curvature. We prove that if H is large (in terms of the scalar curvature of N) then M is properly embedded. The proof follow from two theorems. First, if M is a complete stable immersed H-surface in N and H is large, then M is topologically a sphere. Secondly, a theorem of Meeks, Perez and Ros: limit leaves of CMC-laminations are stable. *To cite this article: H. Rosenberg, C. R. Acad. Sci. Paris, Ser. I.*

Résumé

Soit M une surface complète de courbure moyenne constante dans N^3, de courbure bornée. On montre que si H est grand (par rapport à la courbure scalaire de N) alors M est proprement plongée. La preuve utilise deux théorèmes. Le premier est que si M est est une surface complète de courbure moyenne constante stable dans N et si H est grand, alors M est topologiquement une sphère. Le second est un théorème de Meeks, Perez et Ros : Les feuilles limites d'une lamination CMC sont stable. *Pour citer cet article : H, Rosenberg C. R. Acad. Sci. Paris, Ser. I.*

1. Introduction

Let N be an orientable homogeneously regular 3-manifold. This means there is some positive R so that the geodesic balls of N of radius R, centered at any point of N are embedded, and the sectional curvatures of N are bounded by a constant independent of the point of N. Let S denote the scalar curvature function of N. We will prove:

Theorem 1.1 *Let $c > 0$ and H be constants satisfying*

$$3H^2 + S(x) \geqslant c,$$

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Then a complete embedded H-surface M in N, of bounded curvature, is properly embedded.

Remark 1 A hypothesis on the size of H is necessary; consider a foliation of a flat 3-torus by dense planes ($H = 0$). For $H \neq 0$, the above theorem applies in the flat 3-torus.

The proof is an application of two theorems (more precisely, of one theorem and the proof of the other theorem). The author proves in [2], the following theorem.

Theorem 1.2 *Let N be a homogeneously regular 3-manifold and let M be a complete surface immersed in N of constant mean curvature H. Suppose M is a (strongly) stable H-surface and*

$$3H^2 + S(x) \geqslant c > 0.$$

Then M is topologically a sphere.

Remark 2 Theorem 1.2 is an immediate consequence of the proof of theorem 1 of [2].

We now explain this. The exact statement of theorem 1 of [2], is that for surfaces M as in theorem 1.2, and allowing ∂M to be non empty (until now, M complete meant empty boundary. When the boundary of M is non empty, then complete means a geodesic is infinitely extendable in both directions or goes to the boundary), then for $p \in M$,

$$\mathrm{dist}_M(p, \partial M) \leqslant \frac{2\pi}{\sqrt{3c}}.$$

In particular, when $\partial M = \emptyset$, then M is compact. The proof of theorem 1 of [2] involves studying the metric $d\tilde{s}^2 = u ds^2$, where ds^2 is the metric on M and u is a Jacobi function on M coming from stability, i.e., $u > 0$ is a solution of the linearized operator

$$L(f) = 0;$$

where

$$L = \Delta + |A|^2 + \mathrm{Ric}(n),$$

where Δ is the Laplacian of the metric M, A the second fundamental form, and n a unit normal to M. It is shown in [2], that the metric $d\tilde{s}^2$ has positive intrinsic curvature, so M is a sphere when M is compact.

The second theorem we need for the proof of theorem 1.1, says that a limit leaf of a CMC lamination, is (strongly) stable [1].

Now we can prove theorem 1.1. If M is not properly embedded, then the closure of M is a CMC lamination with a limit leaf M_0, and M_0 is stable [1]. By theorem 1.2, M_0 is a sphere.

Now M spirals towards M_0 ($M_0 \subset \bar{M}$ and $M_0 \neq M$), so one can lift paths on M_0 into paths of M. Since M_0 is compact and simply connected, lifting paths defines an immersion of M_0 into M.

More precisely, for each point $p \in M_0$, there is a disk neighborhood B of p contained in M_0, and there exist pairwise disjoint disk neighborhoods $B(n)$ in M that converge uniformly to B in the normal bundle to B. Thus any path in B lifts to a path in $B(n)$ for n large. Now cover M_0 by a finite number of such disk neighborhoods B. Any compact path of M_0 starting at p, then lifts to a path in M, by lifting to the $B(n)$ for n large. More precisely, the path is covered by a finite number of disk neigborhoods B. Then one lifts each arc of the path to a nearby path in M, and one continues by lifting the arc in the adjacent disc. M_0 simply connected implies the end point of the lifting does not depend on the path of M_0 chosen. Thus one can lift M_0 into a nearby part of M.

The image of M_0 is open and closed in M so M is compact; a contradiction. This proves theorem 1.1.

We remark that M_0 may not be embedded, there may be isolated point of M_0 where two sheets of M_0 touch on their mean concave sides. But this poses no obstacle to lifting paths of M_0 to M.

Remark 3 Suppose N is a compact 3-manifold with positive Ricci curvature. Then if M is a complete embedded H-surface in N, of bounded curvature, we have M is compact. Otherwise the lamination $\bar{M}$ would have a stable leaf M_0 by the theorem of Meeks, Perez and Ros. Then M_0 is a compact sphere by theorem 1.2. Stability means $-\int_{M_0} fL(f) \geqslant 0$, for all f of compact support on M_0. Take $f = 1$ on M_0, then

$$L(f) = \Delta f + (|A|^2 + \mathrm{Ric(n)}) > 0;$$

a contradiction.

References

[1] W. Meeks, J. Perez, A. Ros, *Limit leaves of a CMC lamination are stable*, arxiv, 0801.4345v1[mathDG],28 Janv 2008.

[2] H. Rosenberg, *Constant mean curvature surfaces in homogeneously regular 3-manifolds*, bull. Austral. Math. Soc., vol 74(2006),227-238.