The Dirichlet problem for CMC surfaces in Heisenberg space

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We study constant mean curvature graphs in the Riemannian 3-dimensional Heisenberg spaces $\mathcal{H} = \mathcal{H}(\tau)$. Each such $\mathcal{H}$ is the total space of a Riemannian submersion onto the Euclidean plane $\mathbb{R}^2$ with geodesic fibers the orbits of a Killing field. We prove the existence and uniqueness of CMC graphs in $\mathcal{H}$ with respect to the Riemannian submersion over certain domains $\Omega \subset \mathbb{R}^2$ taking on prescribed boundary values.

1 Introduction

In recent years, there has been much research on minimal and constant mean curvature surfaces (CMC) in the simply connected homogeneous 3-manifolds, other than space forms. Figueroa, Mercuri and Pedrosa [5] gave many interesting such surfaces in $\mathcal{H}$, each invariant by Killing vector fields of the ambient space. Daniel [4] and Abresch-Rosenberg [1], [2] have also obtained some interesting results on these surfaces. For example, the latter authors proved that the only immersed H-surfaces in $\mathcal{H}$ which are homeomorphic to the 2-sphere are precisely the rotational H-spheres. We mention that the classical Alexandrov Theorem is not yet known in $\mathcal{H}$: "Is a compact embedded H-surface a rotational sphere".

It is natural (and we believe important) to solve the Dirichlet problem in $\mathcal{H}$; we do this here.

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2 Preliminaries

2.1 The Heisenberg space

Let $\mathcal{H}$ denote the three-dimensional Heisenberg Lie group endowed with a left invariant metric. In fact, we have a one-parameter family of metrics indexed by bundle curvature by a real parameter $\tau \neq 0$. The spaces are simply connected homogeneous Riemannian manifolds carrying a 4-dimensional isometry group. In global exponential coordinates they are $\mathbb{R}^3$ endowed in standard coordinates with the metrics

$$ds^2 = dx^2 + dy^2 + (\tau(ydx - xdy) + dz)^2.$$

A global orthonormal tangent frame is given by

$$E_1 = \partial_x - \tau y \partial_z, \quad E_2 = \partial_y + \tau x \partial_z, \quad E_3 = \partial_z.$$

The corresponding Riemannian connection is $\bar{\nabla}_{E_j} E_j = 0$, $1 \leqslant j \leqslant 3$, and

$$\bar{\nabla}_{E_1} E_3 = \bar{\nabla}_{E_3} E_1 = -\tau E_2, \quad \bar{\nabla}_{E_2} E_3 = \bar{\nabla}_{E_3} E_2 = \tau E_1$$

$$\bar{\nabla}_{E_1} E_2 = -\bar{\nabla}_{E_2} E_1 = \tau E_3.$$

In particular,

$$[E_1, E_2] = 2\tau E_3 \quad \text{and} \quad [E_1, E_3] = 0 = [E_2, E_3].$$

The Heisenberg space is a Riemannian submersion $\pi\colon \mathcal{H} \to \mathbb{R}^2$ over the standard flat Euclidean plane $\mathbb{R}^2$ whose fibers are the vertical lines. Thus the fibers are the trajectories of a unit Killing vector field and hence geodesics. The horizontal vector fields E_1, E_2 are basic since they are the horizontal lifts of the vector fields of the orthonormal coordinate base of $\mathbb{R}^2$, namely, $\pi_*(E_1) = \partial_x$ and $\pi_*(E_2) = \partial_y$.

The isometries of the space are the translations generated by the Killing vector fields

$$F_1 = \partial_x + \tau y \partial_z, \quad F_2 = \partial_y - \tau x \partial_z, \quad F_3 = \partial_z,$$

and the rotations about the z-axis corresponding to

$$F_4 = -y\partial_x + x\partial_y.$$

The translations corresponding to F_1 and F_2 are, respectively,

$$(x, y, z) \mapsto (x + t, y, z + \tau t y)$$

and

$$(x, y, z) \mapsto (x, y + t, z - \tau t x)$$

where $t \in \mathbb{R}$. Thus, by the group of isometries vertical planes go to vertical planes, and Euclidean lines go to Euclidean lines. For additional information, we refer to [4].

2.2 Graphs

We denote by $S_0 \subset \mathcal{H}$ the surface whose points satisfy $z = 0$. Given a domain $\Omega \subset \mathbb{R}^2$ throughout the paper we also denote by Ω its lift to S_0. We define the *graph* $\Sigma(u)$ of $u \in C^0(\bar{\Omega})$ on Ω as

$$\Sigma(u) = \{(x, y, u(x, y)) \in \mathcal{H} : (x, y) \in \Omega\}.$$

Consider the smooth function $u^*\colon \mathcal{H} \to \mathbb{R}$ defined as $u^*(x, y, z) = u(x, y)$ and set $F(x, y, z) = z - u^*(x, y, z)$. Then $\Sigma(u) = F^{-1}(0)$, and therefore

$$2H = \operatorname{div}\left(\frac{\bar{\nabla} F}{|\bar{\nabla} F|}\right).$$

Here div and $\bar{\nabla}$ denote the divergence and gradient in $\mathcal{H}$ and the mean curvature function H of the graph is with respect to the downward pointing normal vector.

We have

$$\bar{\nabla} F = -(\tau y + u_x)E_1 + (\tau x - u_y)E_2 + E_3.$$

Since E_1, E_2 are basic, using the Riemannian submersion one shows that the H-graph equation is

$$\operatorname{div}_{\mathbb{R}^2}\left(\frac{\alpha}{W}\partial_x + \frac{\beta}{W}\partial_y\right) + 2H = 0 \tag{1}$$

where

$$\alpha = \tau y + u_x, \quad \beta = -\tau x + u_y$$

and

$$W^2 = 1 + \alpha^2 + \beta^2.$$

It follows easily that $\Sigma(u)$ has mean curvature function H if and only if u is a solution of the following PDE

$$Q_H(u) := \frac{1}{W^3}\left((1 + \beta^2)u_{xx} + (1 + \alpha^2)u_{yy} - 2\alpha\beta u_{xy}\right) + 2H = 0 \tag{2}$$

for α, β and W as above. We remark that this is the Euclidean mean curvature equation for $\tau = 0$.

2.3 Cylinders and cones

Let $\gamma\colon I \to S_0 \subset \mathcal{H}$ be a smooth curve parametrized on an interval $I \subset \mathbb{R}$ where S_0 is as above. We assume that $\gamma = \gamma(s)$ is parametrized so that $\bar{\gamma} = \pi \circ \gamma$ carries a parametrization by arc length. Thus $\gamma(s) = (x(s), y(s), 0)$ satisfies $(x')^2 + (y')^2 = 1$.

The *vertical cylinder* $\mathbb{C}_\gamma \subset \mathcal{H}$ over γ is the surface generated by taking through each point of γ the vertical geodesic fiber. Thus $\mathbb{C}_\gamma$ is parametrized by φ: $I \times \mathbb{R} \to \mathcal{H}$ given by

$$\varphi(s,t) = (x(s), y(s), t).$$

Then, the mean curvature $H_{\mathbb{C}}$ (taken to be non-negative) of $\mathbb{C}_\gamma$ is

$$H_{\mathbb{C}}(s) = H_{\mathbb{C}}(s,t) = \frac{k(s)}{2} \tag{3}$$

where $k(s)$ is the geodesic curvature function of $\bar{\gamma}$ with respect to the Euclidean metric. Notice that $H_{\mathbb{C}}$ is independent of the parameter τ. To see that (3) holds, first observe that the horizontal lift T of $\bar{\gamma}\,' = d\bar{\gamma}/ds$ to each point of $\mathbb{C}_\gamma$ forms a horizontal unit tangent vector field. Since $\mathbb{C}_\gamma$ is ruled by vertical geodesics, it follows that the mean curvature of $\mathbb{C}_\gamma$ is $2H_{\mathbb{C}} = \langle \bar{\nabla}_T T, N\rangle$, where N is the Gauss map of the cylinder $\mathbb{C}_\gamma$ chosen so that $H_{\mathbb{C}}$ is non-negative. But N is the horizontal lift of a unit normal vector field η to $\bar{\gamma}$ in $\mathbb{R}^2$, and hence $\langle \bar{\nabla}_T T, N\rangle = \langle D_{\bar{\gamma}'}\bar{\gamma}', \eta\rangle = k$, where D denotes the Euclidean connection.

The *cone* $\mathcal{C}_\gamma \subset \mathcal{H}$ with vertex $P \in \mathcal{H}\backslash S_0$ and base curve γ as above is just the Euclidean cone in $\mathbb{R}^3$ constituted of straight lines from P through points of γ. Thus $\mathcal{C}_\gamma$ is parametrized by

$$\psi(s,t) = (1-t)P + t\gamma(s)$$

where $t \in (0, +\infty)$.

Vertical lines remain invariant under the isometries of $\mathcal{H}$. Thus the same holds for vertical cylinders. Also Euclidean lines are sent to Euclidean lines by isometries of $\mathcal{H}$, and vertical planes as well. Thus cones are also invariant by isometries. Hence, to analyze the behavior of the mean curvature of a cone we may assume that the vertex is $P = (0,0,c)$ where $c \neq 0$. Then, either a computation using (2) or by a direct computation, we obtain that the mean curvature $H = H(s,t)$ of $\mathcal{C}_\gamma$ pointing down is given by

$$H = \frac{ct^2(x^2+y^2+c^2)(y''x' - x''y')}{2(\tau^2 t^4(x^2+y^2)(x'y-y'x)^2 + 2c\tau t^3(xx'+yy')(x'y-y'x) + t^2(c^2+(x'y-y'x)^2)^{3/2}}.$$

Here the sign of H is non-negative when γ is a convex Jordan curve in $\mathbb{R}^2$. In particular,

$$H(s,1) \to H_{\mathbb{C}}(s) \quad \text{as} \quad c \to +\infty.$$

and

$$H(s_0,t) \to +\infty \quad \text{as} \quad t \to 0^+$$

if $y''x' - x''y' > 0$ at $\gamma(s_0)$.

We also have fixing $t = t_0$ and allowing $c \to +\infty$ that

$$2H(s_0,t_0) \to (y''x' - x''y')(s_0),$$

and this is also a proof that the mean curvature of a cylinder is given by (3).

3 The main result

We now state and prove the Dirichlet theorem in Heisenberg space $\mathcal{H}$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^3 boundary $\Gamma = \partial\Omega$ whose curvature function with respect to the inner orientation is $k > 0$. Let H be a constant satisfying $0 \leq 2H < k$ and let $\varphi \in C^0(\Gamma)$ be given. Then there exists a smooth function u satisfying $u|_\Gamma = \varphi$ whose graph $\Sigma(u)$ in $\mathcal{H}$ has constant mean curvature H.*

Moreover, if M is a compact embedded connected surface inside the vertical cylinder $\mathbb{C}_\Gamma$ over Γ with constant mean curvature H, $\partial M = \partial\Sigma(u)$ and the mean curvature vector of M points down, then $M = \Sigma(u)$.

Proof: First suppose that $H = 0$. In this case we prove a more general existence result. In fact, we allow $k \geq 0$, and φ to have a finite number of discontinuities $E \subset \Gamma$, and at each discontinuity, φ has a left and right limit. The Nitsche graph (see [7]) γ of φ is the graph of φ on $\Gamma \setminus E$ together with the vertical segments over each point of E, joining the left and right limits of φ at this point. The Nitsche graph γ is a Jordan curve on the vertical cylinder $\mathbb{C}_\Gamma$ and its vertical projection to Γ is a monotone (constant on the vertical segments) map.

Since $\mathbb{C}_\Gamma$ is mean convex with respect to the inside of $\mathbb{C}_\Gamma$, there is a least area embedded minimal disk Σ inside $\mathbb{C}_\Gamma$ with $\partial\Sigma = \gamma$.

We claim that Σ is a z-graph over Ω and solves the Dirichlet problem as desired. First observe that Σ is nowhere vertical. To see this, suppose $p \in \text{int } \Sigma$ and the tangent plane to Σ at p is vertical. Let $\beta \in \mathbb{R}^2$ be a line such that the vertical plane $P = \pi^{-1}(\beta)$ equals the tangent plane to Σ at p. Then $P \cap \Sigma$ near p is an analytic curve topologically equivalent to $Re(z^k)$, $k \geq 2$, in a neighborhood of $z = 0$. Each branch of these curves leaving p must go to $P \cap \partial\Sigma = P \cap \gamma$, by the maximum principle, i.e., a cycle in (int Σ) $\cap P$ would bound a disk in Σ and we could touch this disk at an interior point with another vertical plane (which is also a minimal surface). Now $P \cap \gamma$ consists of two points of Γ, or one or two vertical segments of γ, by convexity of Γ. Hence, at least two of the branches of $P \cap \Sigma$ leaving p, go to the same point, or vertical segments of γ. This yields a compact cycle $C \subset P \cap \Sigma$. Σ is simply connected so C bounds a disk $D \subset \Sigma$. Using vertical planes in $\mathcal{H}$, we can touch D at an interior point so D would equal this vertical plane; a contradiction. Thus Σ is nowhere vertical in its interior.

Now Σ separates the vertical cylinder over Γ into two components. So Σ can be oriented with the unit normal pointing up in its interior. Then each vertical line over a point in the interior of Ω, intersects Σ in exactly one point, since at two successive points of intersection the normal to Σ would point up and down. This proves Σ is a graph over the interior of Ω.

Now assume that $H \neq 0$ and φ is continuous. We have seen that u must be a solution of the Dirichlet problem

$$\begin{cases} Q_H(u) = 0 \\ u|_\Gamma = \varphi \end{cases} \tag{4}$$

where Q_H was given in (2). To prove the existence part of the theorem, we use the continuity method. We show that the subset

$$Z := \{t \in [0,1] : \exists\, u_t \in C^3(\Omega) \text{ such that } Q_{tH}(u_t) = 0 \text{ and } u_t|_\Gamma = t\varphi\}$$

is nonempty, open and closed in $[0,1]$. We have that Z is not empty since $0 \in Z$; S_0 is a minimal surface in $\mathcal{H}$. Standard arguments from the theory of quasilinear elliptic PDE's presented in [6] give that Z is open (a consequence of the implicit function theorem). Moreover, any solution of $Q_H(u) = 0$ is smooth in Ω. Finally, that Z is closed follows from the theory in [6] once we show that a priori height and gradient estimates exist.

We have from (2) that any Euclidean plane in $\mathbb{R}^3$ is a minimal surface in $\mathcal{H}$. In particular, each leaf of the foliation of isometric surfaces $z = z_0 =$ constant is minimal and diffeomorphic to the base $\mathbb{R}^2$ by the projection of the Riemannian submersion. It follows using the maximal principle that any solution u of (4) satisfies

$$u \geq \min_{\partial\Omega} \varphi.$$

Fix a point $(x_0, y_0, 0) \in \Omega$. Given $z_0 \in \mathbb{R}$, we consider the cone $C(z_0)$ with vertex $P = (x_0, y_0, z_0)$ constituted of straight lines from P through points of the graph of φ over Γ. Then, the piece $C_\varphi(z_0)$ of $C(z_0)$ from P to the graph of φ is contained inside the vertical cylinder over Γ. Notice that $C(z_0)$ is the cone $C_{\hat{\Gamma}}(z_0)$ over $\hat{\Gamma} = C_\Gamma(z_0) \cap S_0$. Clearly, by choosing z_0 such that $|z_0|$ is large enough, the geodesic curvature of $\hat{\Gamma}$ with respect to the Euclidean metric is positive. In fact, the curve converges to Γ as $|z_0| \mapsto \infty$. Therefore, by our previous discussion on the mean curvature of vertical cylinders and cones we have that choosing z_0 large enough, say $z_0 = z_1$, and z_0 small enough, say $z_0 = z_2$, that $C(z_1)$ has mean curvature strictly larger than H everywhere and $C(z_2)$ has negative mean curvature (this cone is going down). By the maximum principle, they are upper and lower barriers for the CMC H-graph equation on Ω. Thus $C(z_1)$ and the above remark concerning planes below the graph of φ provides an a priori height estimate for any solution of the Dirichlet problem (4) depending only on Ω, H and φ, that is,

$$|u|_0 \leqslant C_0(\Omega, H, \varphi).$$

Moreover, the cones also provide the following bound along Γ for the norm of the Euclidean gradient of u

$$|Du| = \sqrt{u_x^2 + u_y^2} \leqslant C_1(\Omega, H, \varphi).$$

The next result uses techniques developed in [3] to show that global estimates of the gradient reduces to the boundary estimates already obtained.

Lemma 2. *Let $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ be a solution of (4). Assume that u is bounded in Ω and that $|Du|$ is bounded in Γ. Then $|Du|$ is bounded in Ω by a constant that depends only on $|u|_0$ and $\sup_\Gamma |Du|$.*

Proof: To estimate $|Du| = \sqrt{u_x^2 + u_y^2}$ in the interior of Ω it suffices to obtain an estimate for $\omega = \sqrt{\alpha^2 + \beta^2}\, e^{Au}$ for some positive constant A to be chosen later. If ω achieves its maximum on Γ then we have the desired bound. Otherwise, ω must reach its maximum at an interior point $p_0 = (x_0, y_0)$ in Ω.

We may choose coordinates of the ambient space such that

$$\beta(p_0) = -\tau x_0 + u_y(p_0) = 0.$$

We denote

$$v = \alpha(p_0) = \tau y_0 + u_x(p_0).$$

The function $\phi = \ln \omega = \ln \sqrt{\alpha^2 + \beta^2} + Au$ also takes a maximum at $p_0 \in \Omega$. That $\phi_x(p_0) = 0$ yields

$$u_{xx}(p_0) = -Avu_x(p_0), \tag{5}$$

and $\phi_y(p_0) = 0$ gives

$$u_{xy}(p_0) = -\tau(Avx_0 + 1). \tag{6}$$

Moreover, from $\phi_{xx}(p_0) \leqslant 0$ we obtain

$$vu_{xxx}(p_0) \leqslant A^2 v^3 u_x(p_0) + A^2 v^2 u_x^2(p_0) - \tau^2(Avx_0 + 2)^2, \tag{7}$$

and $\phi_{yy}(p_0) \leqslant 0$ yields

$$vu_{xyy}(p_0) \leqslant -Av^2 u_{yy}(p_0) + \tau^2 A^2 x_0^2 v^2 - u_{yy}^2(p_0). \tag{8}$$

On the other hand, from (2) and (5) we have

$$u_{yy}(p_0) = -2H(1 + v^2)^{1/2} + \frac{Av}{1 + v^2} u_x(p_0). \tag{9}$$

Taking the derivative of (2) with respect to x and using (5) and (6) yields

$$u_{xxx} + (1 + v^2)u_{xyy} - 2Av^2 u_x u_{yy} - 2\tau^2 v(A^2 x_0^2 v^2 + 3Ax_0 v + 2) - 6AHv^2(1 + v^2)^{1/2} u_x = 0$$

at the point p_0. Multiplying the last equation by v and using (9) and inequalities (7) and (8) we obtain, after a long computation, that

$$\frac{(v - \tau y_0)^2}{1 + v^2} + \tau^2 x_0^2 \leqslant \frac{1}{A^2}(AG_1(v) + G_2(v))$$

where

$$G_1(v) = \frac{2H\tau y_0(1+v^2)^{1/2}}{v} + \frac{P(v)}{v^4}, \quad G_2(v) = -4H^2 + \frac{Q(v)}{v^4}$$

and $\lim_{v\to\infty} P(v)/v^4 = 0 = \lim_{v\to\infty} Q(v)/v^4$. Therefore,

$$\lim_{v\to\infty} G_1(v) = 2H\tau y_0 \quad \text{and} \quad \lim_{v\to\infty} G_2(v) = -4H^2 < 0.$$

It follows that we can choose $A > 0$ such that

$$\frac{(v-\tau y_0)^2}{1+v^2} + \tau^2 x_0^2 \leqslant \frac{1}{2}.$$

This gives an upper bound for v^2, and hence for $\omega = \sqrt{\alpha^2+\beta^2}\; e^{Au}$. This concludes the proof of the Lemma. ∎

Hence Z is closed, and this concludes the proof of the existence part of the Theorem for $0 \leq 2H < k$. Now we prove that the graph $\Sigma = \Sigma(u)$ is unique. Suppose that M is an embedded H-surface inside the vertical cylinder $\mathbb{C}_\Gamma$ over Γ with $\partial M = \partial\Sigma$. Then M separates $\mathbb{C}_\Gamma$ into two components and we assume the mean curvature vector of M points into the lower component. When the mean curvature vector points toward the upper component, our argument will show that M equals the graph of the function u, equal to φ on Γ, with mean curvature H and mean curvature vector pointing toward the upper component.

The mean curvature of the vertical cylinder over Γ is strictly larger than H and the mean curvature vector points inside the cylinder so the interior of M is disjoint from the cylinder by the comparison principle.

Denote by $\Sigma(t)$ the surface Σ translated t by the flow of the Killing field ∂z. Since $\partial\Sigma$ is a z-graph, we have $\partial\Sigma(t) \cap \Sigma(0) = \emptyset$; $\partial\Sigma(0) = \partial\Sigma$. Since M is compact there is a $T > 0$ such that $\Sigma(T) \cap M = \emptyset$.

Now lower $\Sigma(T)$ to Σ by the flow ∂z, letting t go from T to 0. The mean curvature of each $\Sigma(t)$ points down, so there can be no first contact of $\Sigma(t)$ with M for $t > 0$, by the maximum principle. Thus M is below Σ. Now choose $T < 0$ so that $\Sigma(t) \cap M = \emptyset$. Move $\Sigma(T)$ up to Σ by the flow ∂z, letting t go from T to 0. There can be no first contact of $\Sigma(t)$ with M for $t \neq 0$ by the maximum principle (the mean curvature vector of M points toward the downward component). Therefore M is above Σ, and we obtain that $M = \Sigma$. This concludes the proof of the Theorem. ∎

4 A further result

It would be interesting to know if Theorem 1 holds when we allow $2H = k$. In this section we give the following partial answer.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^3 boundary $\Gamma = \partial\Omega$ whose curvature function with respect to the inner orientation is $k > 0$. Let H be a constant satisfying $|\tau|/\sqrt{3} < H \leq k/2$. Then there exists a smooth function u satisfying $u|_\Gamma = 0$ whose graph $\Sigma(u)$ in $\mathcal{H}$ has constant mean curvature H.*

We need a supersolution w defined in a neighborhood of Γ (better than the cones in the preceding section); w is constructed in the following result.

Proposition 4. *Assume that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Q_H(u) = 0$ in Ω, $u|_\Gamma = 0$ and $|u|_0 < M$. If $0 < 2H \leq k$ on Γ, then there is a constant $C = C(H, \Omega, M)$ such that*

$$\sup_\Gamma |Du| \leq C.$$

Proof: Let γ: $[0, \ell] \to \Gamma$ be a parametrization by arc length and let ν stand for the unit normal vector to Γ pointing to Ω. We parametrize a neighborhood U of Γ in Ω by

$$P = P(s,t) = \gamma(s) + t\nu(s) \tag{10}$$

for $(s,t) \in [0,\ell] \times [0,\epsilon]$, where $0 < \epsilon < 1/k(s)$. We compute (1) on U making use of the orthonormal frame

$$P_t = \nu, \quad \frac{1}{\phi} P_s = \gamma'$$

where $\phi(s,t) = 1 - tk(s) > 0$. Notice that (1) can be written as

$$Q_H(u) = \mathrm{div}_{\mathbb{R}^2}\left(\frac{Z}{\sqrt{1+|Z|^2}}\right) + 2H = 0$$

where $Z(p) = -\tau Jp + Du(p)$ and J is the standard complex structure in $\mathbb{R}^2$. Then,

$$W^3 Q_H(u) = W^3 \mathrm{div}_{\mathbb{R}^2}\left(\frac{1}{W} Z\right) + 2HW^3 = -\frac{1}{2}\langle DW^2, Z\rangle + W^2 \mathrm{div}_{\mathbb{R}^2} Z + 2HW^3, \tag{11}$$

where $W^2 = 1 + |Z|^2$.

We compute $W^3 Q_H(w) = 0$ for $w = w(t)$ to be chosen. Then $Dw = w_t P_t$ and

$$W^2 = 1 + |Z|^2 = w_t^2 - 2\theta w_t + A \tag{12}$$

where $\theta = \tau\langle JP, P_t\rangle = \tau\langle\gamma, \gamma'\rangle$ and $A = 1 + \tau^2|\gamma + t\nu|^2$. Moreover,

$$\mathrm{div}_{\mathbb{R}^2} Z = \Delta w = w_{tt} - k_t w_t$$

where

$$k_t(s) = \langle D_{P_s/\phi} P_s/\phi, P_t\rangle,$$

and hence, $k_0(s) = k(s)$. Thus,

$$W^2 \Delta w = w_t^2 w_{tt} - 2\theta w_t w_{tt} - k_t w_t^3 + 2\theta k_t w_t^2 + A w_{tt} - A k_t w_t. \tag{13}$$

Moreover,

$$DW^2 = (2w_t w_{tt} - 2\theta w_{tt} + A_t)P_t + (-2\theta_s w_t + A_s)\phi^{-2} P_s.$$

Using $JP_t = -\phi^{-1} P_s = -\gamma'$ and $\phi^{-1} J P_s = P_t = \nu$, it is easy to see that

$$\frac{1}{2}\langle DW^2, Z\rangle = w_t^2 w_{tt} - 2\theta w_t w_{tt} + \theta^2 w_{tt} + B w_t + C \tag{14}$$

where the functions B and C are bounded on U and do not depend on w or any of its derivatives. It follows from (11), (12), (13) and (14) that

$$W^3 Q_H(w) = 2H(w_t^2 - 2\theta w_t + A)^{3/2} - k_t w_t^3 + 2\theta k_t w_t^2 + (A - \theta^2) w_{tt} - (A k_t + B) w_t - C.$$

For positive constants L and K choose

$$w(t) = L \ln(1 + K^2 t).$$

Then $w(0) = 0$ and $w_{tt} = -w_t^2/L$. Given $M > 0$ choose $L = M/\ln(1+K)$. Thus,

$$w(t) = \frac{M}{\ln(1+K)} \ln(1 + K^2 t).$$

Hence,

$$w(1/K) = M$$

and

$$w_t(0) = \frac{MK^2}{\ln(1+K)}.$$

We claim that we can choose $K > 1/\epsilon$ large enough such that $Q_H(w) < 0$ for all $(s,t) \in [0,\ell] \times [0, 1/K]$. This fact, together with $w(1/K) = M$ (recall that $|u|_0 < M$) allows us to use w as a barrier from above for Q_H and conclude the proof.

It suffices to show that $Q_H(w) < 0$ at $t = 0$ for K large enough. Since $w_t(0) \to +\infty$ as $K \to +\infty$, the claim is clear at points of Γ where $2H < k$. If $2H = k$ first observe that at $t = 0$

$$\lim_{K\to+\infty} \frac{(w_t^2 - 2\theta w_t + A)^{3/2} - w_t^3 + 2\theta w_t^2}{w_t^2} = -\theta.$$

Then, we have that

$$(A - \theta^2) w_{tt}(0) = -\frac{1}{L}(1 + \tau^2(|\gamma|^2 - \langle\gamma, \gamma'\rangle^2)) w_t^2(0) < 0,$$

and the claim follows from the fact that $L \to 0^+$ as $K \to +\infty$. ∎

Proof of Theorem 3: Let $\Omega(n)$ be the domain with boundary

$$P(s, 1/n) = \gamma(s) + \frac{1}{n}\nu(s)$$

for large n, so $\partial\Omega(n)$ is smooth. By Theorem 1 there exists an H-graph $\Sigma(n)$ with $\partial\Sigma(n) = \partial\Omega(n)$, since the curvature of $\partial\Sigma(n)$ is strictly greater than $2H$. Let u_n be the function with graph $\Sigma(n)$

The curvature tensor of $\mathcal{H}$ is given for any $X, Y, Z \in T\mathcal{H}$ by

$$R(X, Y)Z = -3\tau^2(X \wedge Y)Z + 4\tau^2 R_1(\partial_z; X, Y)Z$$

where

$$R_1(\partial_z; X, Y)Z = \langle Y, Z\rangle\langle X, \partial_z\rangle\partial_z + \langle Y, \partial_z\rangle\langle Z, \partial_z\rangle X - \langle X, Z\rangle\langle Y, \partial_z\rangle\partial_z - \langle X, \partial_z\rangle\langle Z, \partial_z\rangle Y.$$

Thus the (not normalized) scalar curvature of $\mathcal{H}$ is $S = -\tau^2$.

By Theorem 1 of [8], there is a positive constant L such that $|u_n|_0 \leq L$ for each n. By the maximum principle, $u_{n+1} > u_n$ on the domain of u_n. Since the u_n are uniformly bounded by L, the function

$$u(x) = \lim_{n\to\infty} u_n(x),$$

is well defined for $x \in \Omega$ and is an H-graph in Ω. Moreover, the upper barrier w constructed in Proposition 4 shows that u takes the value zero on the boundary. ∎

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