Khintchine type inequalities and approximation properties

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Concentration week on

Approximation properties of discrete groups and Operator spaces

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Haagerup's free Khintchine inequalities

We first review Haagerup's theorem on the approximation property on free groups. All results in this first part are due to Haagerup (Invent. Math.; 1979).

- $\mathbb{F}$: a free group on r generators $g_1, ..., g_r$.
- W_d: the subset of $\mathbb{F}$ of words of length d.
- Then $\forall \{x_g\} \subset \mathbb{C}$

$$\Big(\sum_{g\in W_d} |x_g|^2 \Big)^{\frac{1}{2}} \leqslant \Big\| \sum_{g\in W_d} x_g \lambda(g) \Big\|_{C^*_\lambda(\mathbb{F})} \leqslant (d+1) \Big(\sum_{g\in W_d} |x_g|^2 \Big)^{\frac{1}{2}} .$$

Remark. Regarding $\{\lambda(g)\}_{g\in W_d}$ as a lacunary set, this inequality can be interpreted as a Khintchine type inequality in L_∞. Such a phenomenon can never occur in the commutative case: there is no infinite lacunary set in an abelian group.

Projections onto homogeneous subspaces

Let $C_\lambda^*(\mathbb{F})_d \subset C_\lambda^*(\mathbb{F})$ be the subspace spanned by words of length d, i.e. $C_\lambda^*(\mathbb{F})_d$ is the closure of the family of homogeneous polynomials of degree d.
Let $\mathcal{P}_d : C_\lambda^*(\mathbb{F}) \to C_\lambda^*(\mathbb{F})_d$ be the natural projection:

$$\mathcal{P}_d : \sum_{g\in\mathbb{F}} x_g\lambda(g) \mapsto \sum_{g\in W_d} x_g\lambda(g).$$

Corollary. $\|\mathcal{P}_d\| \leqslant d+1$.

Proof.

$$\Big\| \sum_{g\in W_d} x_g\lambda(g)\Big\| \leqslant (d+1)\Big(\sum_{g\in W_d} |x_g|^2\Big)^{\frac{1}{2}}$$

$$\leqslant (d+1)\Big(\sum_{g\in\mathbb{F}} |x_g|^2\Big)^{\frac{1}{2}} = (d+1)\Big\| \sum_{g\in W_d} x_g\lambda(g)\Big\|.$$

Poisson semigroup

- Let $t > 0$. Then the function $g \mapsto e^{-t|g|}$ is a positive definite function on $\mathbb{F}$.
- Consequently, this function induces a completely positive Herz-Schur multiplier on $C^*_\lambda(\mathbb{F})$:

$$\mathcal{T}_t : \sum_{g\in\mathbb{F}} x_g \lambda(g) \mapsto \sum_{g\in\mathbb{F}} e^{-t|g|} x_g \lambda(g).$$

- In particular, $\|\mathcal{T}_t\| \leqslant 1$. Also note that

$$\mathcal{T}_t = \sum_{k=0}^{\infty} e^{-tk} \mathcal{P}_k \,.$$

Truncation

Let

$$\mathcal{T}_{t,n} = \sum_{k=0}^{n} e^{-tk}\mathcal{P}_k \quad \text{and} \quad \mathcal{T}_n = \mathcal{T}_{\frac{1}{\sqrt{n}},n}.$$

- $\mathcal{T}_n$'s are finite rank maps (if $\mathbb{F}$ is finitely generated).
- $\lim_n \mathcal{T}_n = \mathrm{Id}_{C^*_\lambda(\mathbb{F})}$ in point-norm topology:

$$\lim_{n\to\infty} \mathcal{T}_n(x) = x, \quad \forall\, x \in C^*_\lambda(\mathbb{F}).$$

- $\lim_{n\to\infty} \|\mathcal{T}_n\| = 1$.
- **Conclusion**: $C^*_\lambda(\mathbb{F})$ has the metric approximation property (MAP).

Proof. We have

$$\|\mathcal{T}_{t,n}\| \leqslant \|\mathcal{T}_t\| + \sum_{k>n} e^{-kt}\|\mathcal{P}_k\| \leqslant 1 + \sum_{k>n}(k+1)e^{-kt} \leqslant 1 + \frac{(n+1)e^{-nt}}{(1-e^{-t})^2}.$$

Fourier algebra and von Neumann algebra

Let $VN(\mathbb{F})$ and $A(\mathbb{F})$ be the group vN algebra and the Fourier algebra of $\mathbb{F}$, respectively. Note that $A(\mathbb{F})^* = VN(\mathbb{F})$. Then

- $\mathcal{T}_n$'s extend to bounded maps on $VN(\mathbb{F})$ and $A(\mathbb{F})$.
- $\lim_n \mathcal{T}_n = \mathrm{Id}_{A(\mathbb{F})}$ in point-norm topology; $\lim_n \mathcal{T}_n = \mathrm{Id}_{VN(\mathbb{F})}$ in point-w* topology, i.e.

$$\lim_n \mathcal{T}_n(x) = x \quad * \text{-weakly}, \quad \forall\, x \in VN(\mathbb{F}).$$

- We still have $\lim_{n\to\infty} \|\mathcal{T}_n\| = 1$ for these extensions.
- **Conclusion**: $A(\mathbb{F})$ and has MAP and $VN(\mathbb{F})$ has w*-MAP.

Completely bounded approximation property (CBAP)

All previous results can be generalized to the vector-valued setting, i.e. to the category of operator spaces.
Recall: An operator space E has the CBAP if there is a net (T_i) of finite rank maps on E such that

$$\sup_i \|T_i\|_{cb} < \infty \quad \text{and} \quad \lim_i T_i(x) = x, \quad \forall\, x \in E.$$

If in addition, $\|T_i\|_{cb} \leqslant 1$ for all i, E has the completely contractive approximation property (CCAP).

- Since $\mathcal{T}_t$ is c.p., $\mathcal{T}_t$ is c.b. and $\|\mathcal{T}_t\|_{cb} = \|\mathcal{T}_t\| = 1$.
- We also have $\|\mathcal{P}_d\|_{cb} \leqslant d + 1$. This can be proved in different ways. One of them is via the vector-valued version of the previous Khintchine inequality (due to Buchholz).
- So as before, $\lim_{n\to\infty} \|\mathcal{T}_n\|_{cb} = 1$.
- **Conclusion**: $A(\mathbb{F})$ and $C_\lambda^*(\mathbb{F})$ have CCAP; $VN(\mathbb{F})$ has w*-CCAP.

In other words, $\mathbb{F}$ is weakly amenable with constant 1.
Recall: a discrete group G is weakly amenable if there is a net of functions $f_i : G \to \mathbb{C}$ with finite support such that f_i converges pointwise to the constant function 1 and such that their associated Herz-Schur multipliers satisfying $\limsup_i \|M_{f_i}\|_{cb} < \infty$. The Haagerup constant $\Lambda(G)$ of G is then defined to be the infimum over all such lim sup.

It is well-known that the following properties are equivalent:

- G is weakly amenable with constant C.
- $C^*_\lambda(G)$ has the CBAP with constant C.
- $VN(G)$ has the w*-CBAP with constant C.

Summary

To prove approximation properties we need:

- Poisson type semigroup $(\mathcal{T}_t)_{t \geqslant 0}$ of (complete) contractive or uniformly bounded maps, which is continuous with $\mathcal{T}_0 = \mathrm{Id}$;
- (completely) bounded truncations to homogeneous parts;
- Khintchine type inequalities for homogeneous parts that can be used to deal with the truncations above.

This happens in many other situations. For instance:

- Coxeter groups (Fendler; 2002).
- Free von Neumann algebras.
- The second part of this talk is to show that this procedure can be applied to reduced free products of C*-algebras or vN algebras. This part is based on a joint work with Ricard (Crelle; 2006).

Reduced free product

- $(A_i, \phi_i)_{i\in I}$ is a family of unital C*-algebras with states ϕ_i whose GNS constructions (π_i, H_i, ξ_i) are faithful.
- $\mathring{H}_i = \xi_i^\perp$ and $\mathring{A}_i = \{a \in A \ : \ \phi_i(a) = 0\}$.
- Fock space $\mathcal{F} = \mathbb{C}\cdot\Omega \ \bigoplus \bigoplus_{\substack{n\geqslant 1 \\ i_1\neq i_2\neq\cdots\neq i_n}} \mathring{H}_{i_1}\otimes\cdots\otimes\mathring{H}_{i_n}$.
- $\mathcal{A} = *_{i\in I}(A_i, \phi_i)$ is the closure in $\mathbb{B}(\mathcal{F})$ of the algebraic free product $A = \mathbb{C}1 \bigoplus \bigoplus_{\substack{d\geqslant 1 \\ i_1\neq i_2\neq\cdots\neq i_d}} \mathring{A}_{i_1}\otimes\cdots\otimes\mathring{A}_{i_d}$
- The states ϕ_i determine a state ϕ on $\mathcal{A}$ given by:

  $$\phi(1) = 1 \quad \text{and} \quad \phi(a_1\otimes\cdots\otimes a_d) = 0$$

  for $d \geqslant 1$, and $a_1\otimes\cdots\otimes a_d \in \mathring{A}_{i_1}\otimes\cdots\otimes\mathring{A}_{i_d}$ with $i_1 \neq \ i_2 \neq \cdots \neq \ i_d$.
- $\mathcal{A}_d$ is the homogeneous part of degree d, the closure of

  $$\bigoplus_{i_1\neq i_2\neq\cdots\neq i_d} \mathring{A}_{i_1}\otimes\cdots\otimes\mathring{A}_{i_d}.$$

Normal order decomposition of homogeneous polynomials

For each $k \in I$, let

$$\mathcal{F}_k = \bigoplus_{\substack{n \geqslant 1 \\ k = i_1 \neq i_2 \neq \cdots \neq i_n}} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}$$

Let P_k be the projection from $\mathcal{F}$ onto $\mathcal{F}_k$. Then for $a \in \mathring{A}_k$

- $P_k^{\perp} a P_k^{\perp} = 0$.
- $P_k a P_k^{\perp}$ acts as a creation operator w.r.t. the natural gradation of $\mathcal{F}$.
- $P_k^{\perp} a P_k$ acts as an annihilation operator.
- $P_k a P_k$ is diagonal.

Using these properties we have the following decomposition that is important for us: For $a_1 \otimes \cdots \otimes a_d \in \mathring{A}_{i_1} \otimes \cdots \otimes \mathring{A}_{i_d}$, with $i_1 \neq i_2 \neq \cdots \neq i_d$

$$\begin{aligned}
& a_1\, a_2\, \dots\, a_d \\
&= \prod_{k=1}^{d} (P_{i_k} + P_{i_k}^{\perp}) a_k (P_{i_k} + P_{i_k}^{\perp}) \\
&= \sum_{k=0}^{d} P_{i_1} a_1 P_{i_1}^{\perp} \cdots P_{i_k} a_k P_{i_k}^{\perp} \cdot \\
&\qquad\qquad P_{i_{k+1}}^{\perp} a_{k+1} P_{i_{k+1}} \cdots P_{i_d}^{\perp} a_d P_{i_d} \\
&+ \sum_{k=0}^{d-1} P_{i_1} a_1 P_{i_1}^{\perp} \cdots P_{i_k} a_k P_{i_k}^{\perp} P_{i_{k+1}} a_{k+1} P_{i_{k+1}} \cdot \\
&\qquad\qquad P_{i_{k+2}}^{\perp} a_{k+2} P_{i_{k+2}} \cdots P_{i_d}^{\perp} a_d P_{i_d}.
\end{aligned}$$

A different operator space structure on $\mathcal{A}_d$

- $L_1 \subset \mathbb{B}(\mathcal{F})$, the operator subspace spanned by $(P_k \mathring{A}_k P_k^\perp)_{k\in I}$. Then

  $$L_1 \approx \Big(\bigoplus_{k\in I} \mathring{H}_k\Big)_C \quad \text{(column structure) completely isometrically.}$$

  More precisely, for $a_{k,i} \in \mathring{A}_k$ and $m_{k,i} \in \mathbb{M}_m$:

  $$\Big\| \sum_{k,i} P_k a_{k,i} P_k^\perp \otimes m_{k,i} \Big\| = \Big\| \sum_k \sum_{i,j} \phi_k(a_{k,i}^* a_{k,j}) m_{k,i}^* m_{k,j} \Big\|^{\frac{1}{2}}.$$

  Moreover, the natural map $\theta_1 : \mathcal{A}_1 \to L_1$ defined by $\theta_1(a) = P_k a P_k^\perp$ if $a \in \mathring{A}_k$ is a complete contraction.
- $K_1 \subset \mathbb{B}(\mathcal{F})$, the operator subspace spanned by $(P_k^\perp \mathring{A}_k P_k)_{k\in I}$. Then

  $$K_1 \approx \Big(\bigoplus_{k\in I} \overline{\mathring{H}_k}\Big)_R \quad \text{(row structure)}$$

  and the map $\rho_1 : \mathcal{A}_1 \to K_1$ defined by $\rho_1(a) = P_k^\perp a P_k$ if $a \in \mathring{A}_k$ is a complete contraction.

Algebraically, we can identify $\mathcal{A}_d$ with a subspace of $\mathcal{A}_1^{\otimes^d}$. Let X_d be the operator space

$$\bigoplus_{k=0}^{d} L_1^k \otimes_h K_1^{d-k} \bigoplus \bigoplus_{k=0}^{d-1} L_1^k \otimes_h \ell_\infty((A_i)) \otimes_h K_1^{d-k-1}$$

where $L_1^k = L_1^{\otimes_h^k}$ and $K_1^k = K_1^{\otimes_h^k}$.

Now we introduce a new operator space structure on $\mathcal{A}_1^{\otimes^d}$ via the following inclusion

$\iota : \mathcal{A}_1^{\otimes^d} \to X_d$ defined by

$$\iota(a) = ((\theta_1^k \otimes \rho_1^{d-k}(a))_{k=0}^{d}, (\theta_1^k \otimes \mathrm{Id} \otimes \rho_1^{d-k-1}(a))_{k=0}^{d-1}).$$

Khintchine inequalities for free chaos

Theorem. We have a complete isomorphism $E_d \approx \mathcal{A}_d$. More precisely, for any $n \geqslant 1$ and $x \in \mathbb{M}_n(\mathcal{A}_d)$, we have :

$$\|\iota(x)\|_{\mathbb{M}_n(E_d)} \leqslant \|x\|_{\mathbb{M}_n(\mathcal{A}_d)} \leqslant (2d+1)\|\iota(x)\|_{\mathbb{M}_n(E_d)}.$$

Remark. In the case $d = 1$, the inequality above reads as: for any $a_i \in \mathring{A}_i$ and $x_i \in \mathbb{M}_n$

$$\Big\| \sum_i a_i \otimes x_i \Big\|_{\mathcal{A} \otimes \mathbb{M}_n} \approx \max \Big\{ \max_i \|a_i\|_{A_i} \|x_i\|_{\mathbb{M}_n} ,$$

$$\Big\| \sum_i \phi_i(a_i^* a_i) x_i^* x_i \Big\|_{\mathbb{M}_n}^{1/2} , \Big\| \sum_i \phi_i(a_i a_i^*) x_i x_i^* \Big\|_{\mathbb{M}_n}^{1/2} \Big\}.$$

This is due to Voiculescu in the scalar case and to Junge in the vector-valued case.

Truncated Poisson maps

Theorem. The natural projection $\mathcal{P}_d$ from $\mathcal{A}$ onto $\mathcal{A}_d$ is c.b. and $\|\mathcal{P}_d\|_{cb} \leqslant \max(4d, 1)$.

Poisson semigroup. For any $t > 0$ let

$$\mathcal{T}_t = \sum_{k=0}^{\infty} e^{-tk} \mathcal{P}_k.$$

Using Blanchard-Dykema's theorem on free product of c.p. maps, one can show that $\mathcal{T}_t$ is a unital c.p. map on $\mathcal{A}$. Thus $\|\mathcal{T}_t\|_{cb} = 1$.

Corollary. Let

$$\mathcal{T}_{t,n} = \sum_{k=0}^{n} e^{-tk} \mathcal{P}_k \quad \text{and} \quad \mathcal{T}_n = \mathcal{T}_{\frac{1}{\sqrt{n}},n}.$$

Then

$$\lim_{n\to\infty} \|\mathcal{T}_n\|_{cb} = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{T}_n = \mathrm{Id}_{\mathcal{A}}\,.$$

Applications to CBAP

- If $\dim A_i < \infty$ for every i, then their reduced free product $\mathcal{A}$ has the CCAP.
- Let (M_i, ϕ_i) be injective von Neumann algebras with normal states ϕ_i with faithful GNS constructions. Then the reduced von Neumann free product $\overline{*}_i(M_i, \phi_i)$ has the w*-CCAP.
  The proof uses again the truncated Poisson maps $\mathcal{T}_n$, Blachard-Dykema's theorem via the following:
  Proposition. Let M be an injective von Neumann algebra and ϕ a normal state on M. Then there are normal finite rank unital c.p. maps preserving ϕ and converging to the identity in the point w*-topology.
- Let $(G_i)_{i \in I}$ be weakly amenable discrete groups with constant 1. Then the free product $\underset{i \in I}{*} G_i$ is also weakly amenable with constant 1.
  This result improves a theorem of Bożejko-Picardello, who proved the same conclusion with the stronger assumption that all G_i are amenable.

Open problems

There are many open problems related to this talk.

1) The most general problem is the following: If all A_i have CCAP, so does $\mathcal{A}$?
   The same problem remains open for vN algebras.
2) We know that reduced von Neumann free products of injective von Neumann algebras have the w*-CCAP. The C*-counterpart of this result is open.
   Namely, if the A_i are nuclear, does $\mathcal{A}$ have CCAP?
3) The previous problem can be reduced to the following: Let A be a nuclear C*-algebra with a state ϕ. Does there exist finite rank unital c.p. maps on A preserving ϕ and converging to Id_A in the point-norm topology?
4) Let $VN_q(H)$ be the vN algebra of Bożejko-Speicher's q-deformation associated to a real Hilbert space H.
   Does $VN_q(H)$ have w*-CBAP for $-1 < q < 1$, $q \neq 0$?
   N.B. $VN_q(H)$ does have w*-MAP.

5) (Shlyakhtenko) Let H be a real Hilbert space and $U = (U_t)_{t \in \mathbb{R}}$ an orthogonal group on H. Let $VN(H, U)$ be Shlyakhtenko's free quasi-free factor associated to (H, U). Does $VN(H, U)$ have w*-CBAP?

   N.B.

   - If U is almost period (i.e. H has an orthonormal basis of eigenvectors of the generator of U), then $VN(H, U)$ is a free product of matrix algebras; so $VN(H, U)$ has CCAP.
   - A khintchine type inequality is available for $VN(H, U)$ (Nou).

6) Let A_i be C*-algebras with QWEP property. Does their reduced free product $\mathcal{A}$ has the same property?
   N.B. OK for the case of vN algebras (Junge; 2006).

Final remark. Exactness is stable under reduced free product of C*-algebras (Dykema; 2004)