Approximation Properties and C^*-algebras

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Sources: 1) Lindenstrauss-Tzafriri: Classical Banach spaces I, Springer
2) Pisier: Introduction to Operator spaces (Part II), LNMS 294 Cambridge
3) Effros/Lance: Tensor products of C^*-algebras, Adv. Math 25 (1977), 1-34.

Approximation property in Banach spaces

Def.: A Banach space X has the approximation property (AP) if for every compact $K \subset X$, $\varepsilon > 0$ there exists a linear finite rank $T : X \to X$ such that

$$\sup_{x \in K} \| Tx - x \| < \varepsilon .$$

Def.: A Banach space X has the BAP if there exists a constant $C > 0$ and a net T_λ of finite rank maps such that

$$\sup_\lambda \| T_\lambda \| \leq C \quad \text{and} \quad \lim_\lambda T_\lambda(x) = x .$$

$\Lambda(X) = \inf\{C\}$. X has MAP if $\Lambda(X) = 1$.

Rem.: (Grothendieck) $K \subset X$ relative compact iff $K \subset \mathrm{absconv}\{x_n : n \in \mathbb{N}\}$ for some sequence with $\lim_n \|x_n\| = 0$.

Examples

- ✎ $X = L_p(\Omega, \Sigma, \mu)$, $1 \leq p \leq \infty$. Take conditional expectation on finite sub σ-algebras.
- ✎ $X = C(K)$. Use Urysohn's Lemma.
- ✎ $X = S_p = \{x \in K(\ell_2) : [tr(|x|^p)]^{1/p} < \infty\}$ has MAP. Use $T_\lambda(x) = p_\lambda x p_\lambda$ such that p_λ has finite rank and $p_\lambda \to 1$ strongly.
- ✎ Also applies for $p = \infty$ and $S_\infty = K(\ell_2)$.

Examples II

Theorem (Enflo) There exists a Banach space without AP.

Theorem (Szankowski) $B(\ell_2)$ does not have AP.

Theorem (Szankowski) There exists $L_p(N)$ (for some N) without BAP.

Theorem (Haagerup) $C_{red}(\mathbb{F}_n)$ has MAP.

Classical results

Def.: $T_\lambda \to_\tau T$ if $\lim_\lambda \sup_{x\in K} \|T_\lambda(x) - T(x)\| = 0$ for all compact K.

Lemma: $(L(X,Y),\tau)^* \ = \ Y^* \otimes_\pi X$.

$X \otimes Y \subset B(X^*, Y^*; C)$, $\varphi_{x\otimes y}(x^*, y^*) = x^*(x)y^*(y)$

$$\|z\|_\pi \ = \ \inf_{z=\sum_k x_k \otimes y_k} \sum_k \|x_k\|\|y_k\| \ .$$

$$\|\sum_k x_k \otimes y_k\|_\varepsilon \ = \ \sup_{\|x^*\|\|y^*\|\leq 1} |\sum_k x^*(x_k)y^*(y_k)| \ .$$

Def.: $\mathcal{F}(X,Y) = \{T : X \to Y : T \text{ finite rank}\}$.

Remark: X has AP iff $\mathcal{F}(X,X)^\tau = L(X,X)$.

Theorem: (Grothendieck) TFAE

i) X has AP;

ii) For all Y: $\overline{\mathcal{F}(Y,X)}^{\tau} = L(Y,X)$;

ii') For all Y: $\overline{\mathcal{F}(X,Y)}^{\tau} = L(X,Y)$;

iii) For all Y: $\overline{\mathcal{F}(Y,X)}^{\|\ \|} = \mathcal{K}(Y,X)$;

iv) $\sum_n \|x_n^*\|\|x_n\| < \infty$, $\forall_x \sum_n x_n^*(x)x_n = 0$, $\Rightarrow \sum_n x_n^*(x_n) = 0$.

Theorem: (Grothendieck)

$$X^* \quad \text{has AP} \quad \Leftrightarrow \quad \forall Y \;\; \overline{\mathcal{F}(X,Y)}^{\|\ \|} = \mathcal{K}(X,Y) \, .$$

Local Reflexivity

Rem: 1) Schauder's theorem says that T is compact iff T^* is compact. 2) Grothendieck used some form of local reflexivity.

Theorem: (Local Reflexivity-weak form) Let $\dim X < \infty$. Then

$$(X \otimes_\varepsilon Y)^{**} \;=\; X \otimes_\varepsilon Y^{**}\,.$$

Proof: Using a δ-net in B_{X^*}, it suffices to consider $X \subset \ell_\infty^m$ with finite m. Then

$$[\ell_\infty^m(Y)]^{**} \;=\; (\ell_1^m(Y^*))^* \;=\; \ell_\infty(Y^{**})\,.$$

Strong Form: Let $F \subset Y^{**}$ and $G \subset Y^*$ finite dimensional. Then there exists $u : F \to Y$ such that

$$\langle u(f), g\rangle \;=\; \langle f, g\rangle \quad f \in F, g \in G \quad \text{and} \quad \|u\|\|u^{-1}\| \;\leq\; (1+\varepsilon)\,.$$

Constant Drop

Theorem: (Grothendieck) $X = Y^*$ and X separable.

$$\text{X has AP} \quad \Rightarrow \quad \text{X has MAP} .$$

Proof: Step 1): X separable: $(Y \otimes_\varepsilon X)^* = Y^* \otimes_\pi X^* = X \otimes_\pi X^*$ (isometrically).

Step 2: Assume $\overline{B_{Y \otimes_\varepsilon X}} \neq B_{L(X,X)}$. Then there exists T_0, φ such that $\varphi(T_0) > 1$ and $|\varphi(T)| \leq 1$ for $T \in Y \otimes_\varepsilon X$. By 1)

$$\varphi(T) \ = \ \langle T, \xi \rangle \quad , \quad \|\xi\|_{X \otimes_\pi X^*} \leq 1 \, .$$

Since X has AP, we have $\varphi(T) = \langle T, \xi \rangle$ for all T. Contradiction for $T = T_0$. ■

Rem.: (Figiel/Johnson) $AP \not\Rightarrow BAP \not\Rightarrow MAP$.

C^*-versions: CB and CP

Def.: $X \subset B(H)$, $Y \subset B(K)$, $u : X \to Y$ is completely bounded if

$$\|u\|_{cb} \;=\; \sup_m \|id_{M_m} \otimes u : M_m(X) \to M_m(Y)\| \;<\infty.$$

Here $M_m(X) \subset M_m(B(H)) = B(\underbrace{H \oplus \cdots \oplus H}_{m \text{ times}})$.

Def.: Let $1 \in X \subset B(H)$ and $X^* = X$. $u : X \to B(H)$ is called completely positive if

$$id_{M_m} \otimes u : M_m(X) \to M_m(B(H))$$

sends positive elements to positive elements.

Def.: u ucp = unital completely positive.

Link: $\|a\| \leq 1 \quad \Leftrightarrow \quad \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \geq 0$. (enough positive elements!)

Cor.: $u : X \to B(H)$ ucp $\Rightarrow$ u completely contractive ($\|u\|_{cb} \leq 1$).

Theorem (Wittstock, Haagerup, Paulsen) $X \subset B(K)$, $u : X \to B(H)$ completely bounded. Then there exists $\pi : B(K) \to B(\mathcal{H})$ such that

$$u(x) \;=\; V\pi(x)W \quad , \quad \|V\|\|W\| \;=\; \|u\|_{cb} \,.$$

Cor.: u ucp $\Leftrightarrow$ $u(1) = 1$ and $\|u\|_{cb} = 1$.

Proof: $1 = VW$, $\|V\| = 1 = \|W\|$. Then $W^* = V$. ∎

CPAP-CBAP

Def.: A subspace $X \subset B(H)$ has the CBAP if there exits a net (T_λ) of finite rank maps such that

$$\lim_\lambda T_\lambda(x) = x \quad \sup_\lambda \| T_\lambda \|_{cb} \le C .$$

$\Lambda(X) = \inf\{C\}$. X CCAP if $\Lambda(X) = 1$.

Def.: A system $1 \in X \subset B(H)$ has the CPAP if there exits a net (T_λ) of finite rank maps ucp maps such that

$$\lim_\lambda T_\lambda(x) = x .$$

Rem.: X CPAP $\Rightarrow$ X CCAP.

First Goal

Theorem: A C^*-algebra. TFAE

i) A nuclear (i.e. $A \otimes_{\min} B = A \otimes_{\max} B$ for all B);

ii) A^{**} injective (i.e. there exits $E : B(H) \to A^{**}$, $E|_{A^{**}} = id$);

iii) A^{**} is semidiscrete (i.e. A^* has CPAP);

iv) There exists $u_\lambda = v_\lambda w_\lambda$, $w_\lambda : A \to M_{n(\lambda)}$, $v_\lambda : M_{n(\lambda)} \to A$ such that $\lim_\lambda u_\lambda(x) = x$ and

$$\sup_\lambda \|v_\lambda\|_{cb} \|w_\lambda\|_{cb} < \infty \, .$$

C^*-norms

Let A and B be unital C^*-algebras. Then $A \otimes B$ is a *-algebra with

$$(a \otimes b)^* = a^* \otimes b^* \quad (a \otimes b)(c \otimes d) = ac \otimes bd .$$

Def.: Let $z = \sum_k a_k \otimes b_k$.

- ✗ $\|z\|_{A \otimes_{\max} B} = \sup\{\|\sigma\pi(z)\|_{B(H)} : \pi, \sigma \text{ both } ^* \text{ homom.} [\pi(a), \sigma(b)] = 0\}$.
- ✗ $A \subset B(H)$, $B \subset B(K)$, $\|z\|_{A \otimes_{\min} B} = \|z\|_{B(H \otimes K)}$.

$$\pi\sigma(z) = \sum_k \pi(a_k)\sigma(b_k)$$

Rem.: The choice of H, K doesn't matter. (Wittstock's theorem).

Prop.: (Effros/Lance) $S(A \otimes_{\max} B) = CP_1(A, B^*) = CP_1(B, A^*)$.

Proof

"$\Leftarrow$": Let $T : A \to B^*$ cp $T(1)(1) = 1$. Define the inner product

$$(\sum_k a_k \otimes b_k, \sum_j c_j \otimes d_j) \; = \; \sum_{j,k} T(a_k^* c_j)(b_k^* d_j) \; .$$

Using GNS construction one finds commuting representations.

"$\Rightarrow$": Let π, σ commuting rep's and $\varphi = \omega_\xi$ a state. Then $T_\varphi(a)(b) = \varphi(a \otimes b)$ is completely positive. Indeed, $a \; \geq \; 0$ and $b \; \geq \; 0$ implies

$$\varphi(a \otimes b) \; = \; (\xi, \sigma(a)^{1/2} \pi(b) \sigma(a)^{1/2} \xi) \; .$$

For completely positivity (i.e. $\sum_{ij} T_\varphi(a_{ij})(b_{ij}) \; \geq \; 0$) we use $\xi_n \; = \; \sum_{i=1}^n e_{ii} \otimes \xi \in \ell_2^{n^2}(H)$ and get

$$\sum_{ij} T_\varphi(a_{ij})(b_{ij}) \; = \; (\xi_n, (id_{M_n} \otimes \sigma)(a)(id_{M_n} \otimes \pi(b))\xi_n) \; \geq \; 0 \; .$$

Prop.: $S(A \otimes_{\min} B) = \overline{\mathcal{F}CP_1(A, B^*)}^{pw-wk^*}$.

Nuclear-Semidiscrete

Let A^* be semidiscrete and T_λ be finite rank and $T_\lambda^*(1)(1) = 1$. Then

$$CP_1(B, A^*) \stackrel{T_\lambda \circ}{\to} \mathcal{F}CP_1(B, A^*) \, .$$

Hence $A \otimes_{\min} B = A \otimes_{\max} B$.

Conversely, let φ be faithful and e support in A^{**}. Let $N = eA^{**}e$ and φ faithful. Let $T_\varphi : N \to N_*^{op} \subset A_{op}^*$ be given by

$$T_\varphi(r)(s) \; = \; (\xi_\varphi, rJsJ\xi_\varphi) \, .$$

Let $T_\lambda \to T_\varphi$ and $\hat{T}_\lambda : N \to N$

$$\hat{T}_\lambda(x) \; = \; T_\lambda(1)^{-1/2} T_\lambda(x) T_\lambda(1)^{-1/2} \text{makes sense} \, .$$

Then T_λ^* gives semidiscrete finite rank approximation.

Min-Max-nuclear

Def.: Let $Y \subset B(H)$ be a subspace and A a C^*-algebra, $z = \sum_k y_k \otimes a_k$

$$\Delta(z) \; = \; \sup\{\| \sum_k \sigma(y_k)\pi(a_k)\| \; : \; \sigma cc \; , \; \pi^* - \text{hom} \; , \; , [\sigma(y), \pi(a)] = 0\} \; .$$

Theorem(Pisier)

$$\begin{aligned} \Delta(z) &= \delta(z) \\ &= \inf_{z=\sum_{ij} y_{ij} \otimes a_i b_j} \|y_{ij}\|_{M_n(Y)} \| \sum_i a_i a_i^* \|^{1/2} \| \sum_j b_j^* b_j \|^{1/2} \; . \end{aligned}$$

Haagerup tensor product+Hahn-Banach, see also Weak*-version by C. LeMerdy/Bogajna.

Trick: Let $Y = X^*$ and $Y \subset B(H)$ such that

$$CB(X, M_n) \;=\; M_n(X^*)\;.$$

Then for $u : X \to A$ finite rank $z_u = \sum_k x_k^* \otimes a_k \in X^* \otimes A$ we have

$$\delta(z) \;=\; \inf_{u=vw} \|w : X \to M_n\|_{cb} \|v : M_n \to A\|_{dec} \;=\; \mathrm{fact}(u)\;.$$

$\|u : B \to A\|_{dec} \;=\; \inf \left\{ \max\{\|T_{11}\|, \|T_{22}\} : \left[\begin{array}{cc} T_{11} & u \\ u^* & T_{22} \end{array} \right] cp \right\}$ is the decomposable norm (linear combination of cp).

δ-norm and max

Theorem: (Pisier) a) $X \subset B(H)$, $u : X \to A$, $\dim(X) < \infty$. Then

$$\sup_B \|u \otimes id_B : X \otimes_{\min} B \to A \otimes_{\max} B\| \ = \ \mathrm{fact}(u)$$

b) $\dim X = \infty$. Then

$$\begin{aligned} &\sup_B \|u \otimes id_B : X \otimes_{\min} B \to A \otimes_{\max} B\| \\ &= \inf_{u = \lim_\lambda v_\lambda w_\lambda} \|w_\lambda : X \to M_{n_\lambda}\|_{cb} \|v_\lambda : M_{n_\lambda} \to A\|_{dec} \\ &=: \ \mathrm{nuc}(u) \ . \end{aligned}$$

Applications of δ-norm

Cor.: $A \otimes_{\min} B = A \otimes_{\max} B \Rightarrow \text{nuc}(id_A) = 1$

Rem.: (JLM) $u : C \to A$ finite rank. Then $\text{fact}(u) = \|u\|_{dec}$.

Exactness

Theorem: (Kirchberg) $C^*(\mathbb{F}_\infty) \otimes_{\min} B(H) = C^*(\mathbb{F}_\infty) \otimes_{\max} B(H)$.

Cor.: A is exact, this means $A \otimes_{\min} (C/I) = A \otimes_{\min} C/A \otimes_{\min} I$, iff the inclusion $\iota : A \to B(H)$ is nuclear.

Proof: "$\Rightarrow$": For $A = M_n$ we have $M_n(C/I) = M_n(C)/M_n(I)$. For $F \subset M_n$ we still have $F \otimes_{\min} (C/I) = F \otimes_{\min} C/F \otimes_{\min} I$ using approximate unit.
"$\Leftarrow$": $B = C^*(F_\infty)/I$ Use

$$B(H) \otimes_{\min} C^*(F_\infty)/B(H) \otimes_{\min} I \;=\; B(H) \otimes_{\max} B\,.$$

Local reflexivity

Let A be a unital C^*-algebras. According to Archbold/Batty A has C iff

$$\forall B \quad A^{**} \otimes_{\min} B^{**} \subset (A \otimes_{\min} B)^{**} \text{ well-defined contraction} .$$

A subspace $X \subset B(H)$ is locally reflexive (property C'') if

$$C''(X) \;=\; \sup_F \|id : X^{**} \otimes_{\min} F \to (X \otimes_{\min} F)\| < \infty .$$

A subspace $X \subset B(H)$ has (property C') if

$$C'(X) \;=\; \sup_B \|id : X \otimes_{\min} B^{**} \to (X \otimes_{\min} B)\| < \infty .$$

Rem.: (Effros/Haagerup) $C'\&C''$ iff C.

Rem.: (J) $C'(X) = ex(X) =: \text{nuc}(\iota_X : X \to B(H))$.

Exactness and WEP

Theorem: (Kirchberg) An exact C^*-algebra is subnuclear and locally reflexive.

Def.: A has WEP (weak expectation property) if $A \subset A^{**} \subset B(H)$ there exists cp $E : B(H) \to A^{**}$ such that $E|_A = id_A$.

Theorem: (Lance) A nuclear iff A locally reflexive and WEP.

LR+WEP $\Rightarrow A^{**}$ injective

We show that A^{**} is injective, i.e. for every $F_1 \subset F_2$ and $u : F_1 \to A^{**}$ there exists $\hat{u} : F_2 \to A$ such that $\hat{u}|_{F_1} = u$ and $\|\hat{u}\|_{cb} = \|u\|_{cb}$:

$$\begin{array}{ccc} F_2 & \stackrel{\hat{u}}{\longrightarrow} & A^{**} \\ \cup & \nearrow_{u} & \\ F_1 & & \end{array}$$

Indeed, $u : F_1 \to A^{**}$ can be approximated by $u_\lambda : F_1 \to A$, pt w^* by LR. Then WEP gives extension $\hat{u}_\lambda$. Limit does the trick.

Note: We have $CP(M_k, A)^{**} = CP(M_k, A^{**})$. But local reflexivity implies

$$CB(M_k, A)^{**} \; = \; CB(M_k, A^{**}) \, .$$

Problem: Characterize C^*-algebras with this property.

A nuclear $\Rightarrow$ WEP

Use $A \to M_{n_\lambda} \to A$ to create $u : A \to \ell_\infty(M_{n_\lambda})$ cpu and cpu $v : \ell_\infty(M_{n_\lambda}) \to A^{**}$ as

$$v((x_\lambda)) \ = \ w^* - \lim_\lambda v_\lambda(x) \in A^{**}$$

(limit only exists in A^{**}) such that $vu = id_A$.

Dual norm

Def.: Let $X \subset B(H)$ and $Y \subset B(H)$. Then the projective norm is defined as

$$\|z\|_{X \hat{\otimes} Y} = \inf_{z=\sum_{ijkl} a_{ik} b_{lj} x_{kl} \otimes y_{ij}} \|(x_{ij})\|_{M_n(X)} \|(y_n)\|_{M_n(Y)} tr(a^* a)^{1/2} tr(b^* b)^{1/2} .$$

Tool: $(X \hat{\otimes} Y)^* = CB(X, Y^*) = CB(Y, X^*)$.

A semidiscrete $\Rightarrow$ LR

Prop.: A semidiscrete. Then $(A \otimes_{\min} F)^* \; = \; A^* \hat{\otimes} F^*$ isometrically.

Proof: Let $u_\lambda : A^* \to A^*$ a net of cp finite rank maps. By JLM (applied to $u_\lambda^* : A^{**} \to A^{**}$), we have a factorization $u_\lambda = v_\lambda w_\lambda$ such that

$$w_\lambda : A^* \to M^*_{n(\lambda)} \quad , \quad v_\lambda : M^*_{n(\lambda)} \to A^* \; .$$

converges point norm. Using $CP(M_n, A^{**}) = CP(M_n, A)^{**}$ we may assume w_λ normal. Let $z \in A^* \otimes F^*$ and λ such that $u_\lambda^* \otimes id(z) \; = \; z$. Then

$$w_\lambda^* \otimes id : M_{n(\lambda)} \otimes_{\min} F \to A \otimes_{\min} F$$

is a contraction and hence

$$\|z\|_{A^* \hat{\otimes} F^*} \; \leq \; \|v_\lambda^*\|_{cb} \|(w_\lambda^* \otimes id)(z)\|_{(M_{n_\lambda} \otimes_{\min} F)^*} \; \leq \; \|w_\lambda\|_{cb} \|v_\lambda\|_{cb} \|z\|_{(A \otimes_{\min} F)^*}$$

Conclusion:

$$(A \otimes_{\min} F)^{**} = [(A \otimes_{\min} F)^*]^* \; = \; (A^* \hat{\otimes} F^*)^* \; = \; A^{**} \otimes_{\min} F \; .$$

Note that for finite dimensional F

$$(X \hat{\otimes} F)^* \; = \; CB(X, F^*) \; = \; X^* \otimes_{\min} F^* \; .$$

Digression QWEP

Def.: B QWEP if $B = A/I$ with A WEP.

Theorem: (Kirchberg) TFAE

i) All B are QWEP;

ii) $C^*(F_\infty) \otimes_{\min} C^*(F_\infty) = C^*(F_\infty) \otimes_{\max} C^*(F_\infty)$;

iii) Every II_1 factor embeds in an ultraproduct R^ω of the hyperfinite II_1 factor.

Remarks

Rem.: B QWEP iff B^{**} QWEP.

Rem.: 1) A WEP (i.e. $A \subset B(H) \xrightarrow{E} A^{**}$. $\Rightarrow A^* \xrightarrow{E^*} B(H)^* \to A^*$.
2) B QWEP iff $id : B^* \overset{\alpha}{\xrightarrow{\text{cpu}}} B(H)^* \overset{\beta}{\xrightarrow{\text{cpu}}} B^*$.
3) Local reflexivity of $B(H)_*$ allows to find factorization

$$id :_{B^*} B^* \overset{\alpha}{\xrightarrow{\text{cpu}}} \prod_{\mathcal{U}} B(H)_* \overset{\beta}{\xrightarrow{\text{cpu}}} B^* \, .$$

Kirchberg: It suffices to have $B^* \subset \prod_{\mathcal{U}} B(H)_*$ isometrically.

4) N vNa. N QWEP iff $N \subset \mathcal{M} \overset{E}{\xrightarrow{\text{normal}}} N$ where $\mathcal{M} = (\prod_{\mathcal{U}} B(H)_*)^*$.

Norms on B^*

Theorem: B is QWEP iff there exists norm $|||\ \ |||$ on $B^* \otimes C^*(\mathbb{F}_\infty)$ such that

i) For all f.d. $G_1, G_2 \subset C^*(\mathbb{F}_\infty)$ and $u : G_1 \to G_2$

$$|||(u \otimes id)(z)||| \leq \|u\|_{cb} |||z||| \ .$$

ii) Let $F \subset M_n$ and $F^* \subset C^*(\mathbb{F}_\infty)$. Then $|||z||| = \|z\|_{(B \otimes_{\min} F)^*}$, for all $z \in B^* \otimes F^*$.

Candidate: $|||z||| = \inf_{z=(a\otimes 1)y(b\otimes 1)} \|a\|_2 \|y\|_{B_{op}^{**} \otimes_{\max} C^*(\mathbb{F}_\infty)} \|b\|_2$.

Rem.: All B QWEP if for every $G \subset C^*(\mathbb{F}_\infty)$ and $u : G \to C^*(\mathbb{F}_\infty)$

$$\inf_{\tilde{u}|_G} \|\tilde{u}\|_{dec} \leq \|u\|_{cb} \ .$$

Connes criterion for injectivity

Theorem: (Connes-Pisier) N is injective if there exists a constant such that

$$\|\sum_i x_i \otimes \bar{x}_i\|_{B(H)\otimes_{\min} \bar{B(H)}} \;\leq\; 1$$

implies $x_i = a_i + b_i$, $a_i \in N$ and $b_i \in N$ and

$$\|\sum_i a_i a_i^*\|^{1/2} + \|\sum_i b_i^* b_i\|^{1/2} \;\leq\; C\,.$$

Remark: There is a space OH with basis e_i such that for $u(e_i) = x_i$

$$\|\sum_i x_i \otimes \bar{x}_i\|_{B(H)\otimes_{\min} \bar{B(H)}} \;=\; \|u : OH \to B(H)\|_{cb}^2\,.$$

And the result is true for $B(H)$.

injective = hyperfinite

Theorem: (Connes) A von Neumann algebra N is injective iff N is hyperfinite, i.e $N = \overline{\bigcup_{A \subset N, \dim A < \infty} A}^{STOP}$.

Note: Moreover, $N_* = \bigcup_\lambda A_\lambda^*$ (enough conditional expectations onto finite von Neumann subalgebra.

Cor.: A^{**} injective $\Rightarrow A^*$ CPAP.

Factor through $M_n \Rightarrow$ nuclear

Assume $u_\lambda = v_\lambda w_\lambda : A \to A$ such that

$$\sup_\lambda \|w_\lambda : A \to M_{n_\lambda}\|_{cb} \|v_\lambda : M_{n_\lambda} \to A\|_{cb} \ \leq \ C \ .$$

Then $ex(A) < \infty$. Hence $ex(A) = 1$. Then A is locally reflexive (Kirchberg). Let $u : OH \to N = A^{**}$ such that $\|u\|_{cb} \leq 1$. By local reflexivity $u = \lim_\lambda u_\lambda : OH \to A$ with $\|u_\lambda\|_{cb} \leq 1$. By factorization

$$u(e_i) \ = \ a_i + b_i \in N$$

such that $\| \sum_i a_i^* a_i \|^{1/2} + \| \sum_i b_i^* b_i \|^{1/2} \ \leq \ C$. Connes' theory applies. ■

Haagerup characterization of WEP

Theorem: (Haagerup) A has WEP iff

$$\|\sum_i x_i \otimes \bar{x}_i\|_{A \otimes_{\min} \bar{A}} = \|\sum_i x_i \otimes \bar{x}_i\|_{A \otimes_{\max} \bar{A}} .$$

Comment: Weaker form of semidiscrete

Rem.: Assume that N_* admits a factorization $u_\lambda : N_* \to N_*$ with

$u_\lambda = w_\lambda v_\lambda$

$$\sup_\lambda \|v_\lambda : N_* \to M_{n_\lambda}^*\|_{cb} \| \|w_\lambda : M_{n_\lambda}^* \to N_*\|_{cb} \leq C .$$

Then it is easy to see that for fd $F_1 \subset F_2$ and $z \in N_* \otimes F_1$

$$\|z\|_{N_* \hat{\otimes} F_1} \leq C\|z\|_{N_* \hat{\otimes} F_2} .$$

Hence $N_* \hat{\otimes} N \subset N_* \hat{\otimes} B(H)$ is a C-isomorphic inclusion. Since $\varphi(n^* \otimes n) = n(n^*)$ has norm ≤ 1, the Hahn-Banach theorem yields

$$\psi \in (N_* \hat{\otimes} B(H))^* = CB(B(H), N)$$

extending $id : N \to N$. This means N is C-injective, hence injective.

Notes: 1) $VN(\mathbb{F}_n)_*$ has CCAP, but does not allow factorization through M_n^*. This means Finite rank cp maps are "self-improving", but cb maps are not.

2) We have seen in Xu's talk that the operator space structure of the space of homogenous polynomials plays an important role (see also Houdayer's talk). This line of work started from Haagerup/Pisier's paper on $\mathrm{span}\{g_i^{\pm 1} : g_i \text{ generator }\}$.

3) In free products the span of the words of length n plays a similar role (Ricard/Xu), but is slightly more involved (additional ℓ_∞ term).

4) Popa and Qzawa proved that wreath product $G = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ is a group such $VN(G)$ has the Haagerup property (DeCornulier/Stalder/Valette) but not the CCAP. Moreover, cool applications to rigidity!