# ON THE GEOMETRY OF COAMOEBAS OF COMPLEX ALGEBRAIC HYPERSURFACES

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Amoeba and coamoeba are a very fascinating notions in mathematics where the first terminology has been introduced by I. M. Gelfand, M M. Kapranov and A. V. Zelevinsky in their book (see [GKZ-94]) in 1994, and the second one by M. Passare and A. Tsikh in 2001. Amoebas (resp. coamoebas) have their spines, contours and tentacles (resp. spines, contours and extra-pieces), and they have many applications in real algebraic geometry , complex analysis, mirror symmetry, algebraic statistics and in several other areas (see [M1-00], [M2-02], [M3-04], [RST-05], [FPT-00], [PR1-04], and [PS-04]). Amoebas and coamoebas are linked in a natural way to the geometry of Newton polytopes, which can be seen in particular with the Viro patchworking principle (i.e., tropical localization) based on the combinatorics of subdivisions of convex lattice polytopes. The purpose of this talk is to describe the relations and the similarities which exist between amoebas and coamoebas of a complex algebraic hypersurfaces. Let $V \subset (\mathbb{C}^*)^n$ be a complex algebraic hypersurface defined by a polynomial $f$ with Newton polytope $\Delta$. The amoeba $\mathscr{A}$ of an algebraic set $V = \{f(z) = 0\}$ in the algebraic torus $(\mathbb{C}^*)^n$ is defined as its image under the mapping $\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log \mid z_1 \mid, \ldots, \log \mid z_n \mid)$. The amoeba's complement has a finite number of convex connected components, corresponding to domains of convergence of the Laurent series expansions of the rational function $\frac{1}{f}$. We know that the spine $\Gamma$ of the amoeba $\mathscr{A}$ has a structure of a tropical hypersurface in $\mathbb{R}^n$ (proved by M. Passare and H. Rullgård in 2000 [PR1-04], and independently by G. Mikhalkin in 2000). In addition the spine of the amoeba is dual to some coherent (i.e. convex) subdivision $\tau$ of the integer convex polytope $\Delta$. It is shown by M. Forsberg, M. Passare and A Tsikh that the set of vertices of $\tau$ is in bijection with the set of complement components of $\mathscr{A}$ in $\mathbb{R}^n$ [FPT-00]. An amoeba is called *solid* if the number of its complement components in $\mathbb{R}^n$ is minimal i.e., equal to the number of the Newton polytope vertices.

The coamoeba $co\mathscr{A}$ of an algebraic set $V = \{f(z) = 0\}$ in $(\mathbb{C}^*)^n$ is defined as its image under the argument mapping $\mathrm{Arg} : (z_1, \ldots, z_n) \mapsto (e^{i\arg(z_1)}, \ldots, e^{i\arg(z_n)})$. It is shown in [N2-08] that the complement components of the closure in the flat torus of the coamoeba of a complex algebraic hypersurface defined by a polynomial $f$ with Newton polytope $\Delta$ are convex and their number don't exceed $n!\,\mathrm{Vol}(\Delta)$. Using geometric properties of coamoebas we prove the famous Passare-Rullgård's conjecture:

**Theorem 1** ([N1-07]). *Let $V_f$ be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a maximally sparse polynomial $f$ (this means its only monomials are those of index in the vertices of its Newton polytope). Then the amoeba $\mathscr{A}_f$ of $V_f$ is solid.*

**Theorem 2** ([N2-08])**.** *Let $V_f$ be an algebraic hypersurface in $(\mathbb{C}^*)^n$ defined by a polynomial $f$ with Newton polytope $\Delta$ and we denote by $co\mathscr{A}$ the image of $V_f$ under the argument map* Arg. *Then we have:*

(a) *The interior of any connected component of $(S^1)^n \setminus co\mathscr{A}$ is a convex set,*

(b) *the number of connected components of $(S^1)^n \setminus \overline{co\mathscr{A}}$ is not greater than $n! \operatorname{Vol}(\Delta)$ where $\overline{co\mathscr{A}}$ is the closure of $co\mathscr{A}$ in the flat torus $(S^1)^n$.*

**Theorem 3** ([N3-08])**.** *Let $V$ be a complex algebraic hypersurface defined by a polynomial $f$ and $co\mathscr{A}$ its coamoeba. Then there exists a continuous deformation of the coamoeba $co\mathscr{A}$ into the coamoeba $co\mathscr{A}_\infty$ of a complex tropical hypersurface $V_\infty$, such that the closure in the torus of the two coamoebas have the same topology (i.e., homeomorphic).*

This means that the coamoeba of a complex algebraic hypersurface has the same topology (i.e., homeomorphic) of the coamoeba of a complex tropical hypersurface; more precisely, their closure in the real torus have the same topology.

The coamoebas of a complex algebraic plane curves have a similar properties than their amoebas, and we have the following (see [MR-00] for the amoebas):

**Theorem 4** ([N4-08])**.** *Let $V$ be a complex algebraic plane curve defined by a polynomial with Newton polygon $\Delta$. Then the area counted with multiplicity of the coamoeba of $V$ cannot exceed $2\pi^2 \operatorname{Area}(\Delta)$, and we have the following equivalent statements:*

(i) $\operatorname{Area}_{mult}(co\mathscr{A}) = 2\pi^2 \operatorname{Area}(\Delta)$,

(ii) *The curve $V$ is real up to multiplication by a constant in $\mathbb{C}^*$, and its real part $\mathbb{R}V$ is a Harnack curve possibly with ordinary real isolated double points.*

Let us give a brief description of our ideas without technical details. Let $A$ be the support of the polynomial $f$. The main ingredients in the construction are a special deformation of the standard complex structure on $(\mathbb{C}^*)^n$, Viro's tropical localization, and Kapranov's theorem [K-00]. We construct a family of polynomials $f_t$ with $0 < t \leq \frac{1}{e}$ such that $f_{\frac{1}{e}} = f$, and we consider the family of the $J_t$-holomorphic hypersurfaces $H_t(\{f_t(z) = 0\})$ where $H_t$ is a self-diffeomorphism of $(\mathbb{C}^*)^n$ conserving the arguments. When $t$ tends to zero, we obtain a complex tropical hypersurface $V_\infty$, such that its coamoeba is a deformation of the coamoeba of $V$ which conserve the same topology. In addition, using the subdivision $\tau$ of $\Delta$ dual to the spine of the amoeba of $V$ and Kapranov's theorem [K-00], we have an algorithm giving an explicit description of the coamoeba of $V_\infty$. In other word, the results are obtained by deformation of the complex structure on the hypersurface to a degenerate structure called complex tropical structure which is a piecewise-linear polyhedral complex in $\mathbb{R}^n$ supplied with some lifting to $(\mathbb{C}^*)^n$ (see G. Mikhalkin [M1-00] and [M2-02]).

We give an example of a polygon which can be never the Newton polygon of a real algebraic plane curve (I mean defined over $\mathbb{R}$) realizing the maximum number of complement components of the coamoeba, but this maximum can be realized by a complex plane curve.

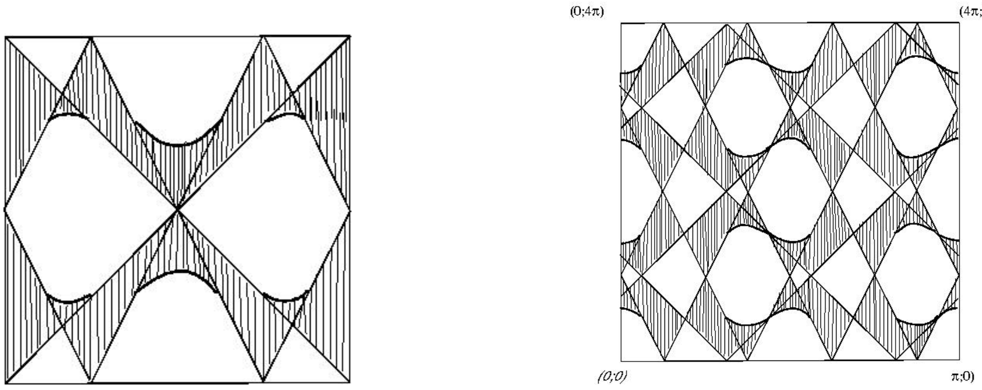

Figure 1. On the left the coamoeba of a real algebraic curve (with five complement components) and on the right the coamoeba of a complex algebraic curve with the same Newton polygon as the first one (with six complement components).

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