Math. Ann. (2008) 340:569–624
DOI 10.1007/s00208-007-0160-9

Mathematische Annalen

Eta-invariants, torsion forms and flat vector bundles

Xiaonan Ma · Weiping Zhang

Received: 25 December 2005 / Revised: 11 July 2007 / Published online: 28 August 2007

Abstract We present a new proof, as well as a $\mathbb{C}/\mathbb{Q}$ extension (and also certain $\mathbb{C}/\mathbb{Z}$ extension), of the Riemann–Roch–Grothendieck theorem of Bismut–Lott for flat vector bundles. The main techniques used are the computations of the adiabatic limits of η-invariants associated to the so-called sub-signature operators. We further show that the Bismut–Lott analytic torsion form can be derived naturally from transgressions of η-forms appearing in the adiabatic limit computations.

1 Introduction

Let M be a compact smooth manifold. For any complex flat vector bundle F over M with the flat connection ∇^F, one can define a mod $\mathbb{Q}$ version of the Cheeger–Chern–Simons character $CCS(F, \nabla^F)$ (cf. [19]) as follows. By a result of Atiyah–Hirzebruch [1], there exists a positive integer k such that kF is a topologically trivial complex vector bundle. Let ∇_0^{kF} be a trivial connection on kF, which can be determined by choosing a global basis of kF. Let $k\nabla^F$ be the connection on kF obtained from the direct sum of k copies of ∇^F. Then we define the mod $\mathbb{Q}$ version of the Cheeger–Chern–Simons character as

$$CCS\left(F, \nabla^F\right) = \frac{1}{k} CS\left(\nabla_0^{kF}, k\nabla^F\right), \tag{1.1}$$

X. Ma
Centre de Mathématiques Laurent Schwartz, UMR 7640 du CNRS, École Polytechnique,
91128 Palaiseau Cedex, France
e-mail: ma@math.polytechnique.fr

W. Zhang (✉)
Chern Institute of Mathematics & LPMC, Nankai University,
Tianjin 300071, People's Republic of China
e-mail: weiping@nankai.edu.cn

where $CS(\nabla_0^{kF}, \nabla^{kF})$ is the Chern-Simons class associated to $(kF, \nabla^{kF}, \nabla_0^{kF})$. It determines a well-defined element in $H^{odd}(M, \mathbb{C}/\mathbb{Q})$ (See Sect. 2.4 for more details).

Let $Z \to M \to B$ be a fibered manifold with compact base and fibers. Let $e(TZ)$ be the Euler class of the vertical tangent vector bundle TZ. The flat vector bundle (F, ∇^F) over M induces canonically a $\mathbb{Z}$-graded flat vector bundle $H^\bullet(Z, F|_Z) = \bigoplus_{i=0}^{\dim Z} H^i(Z, F|_Z)$ over B (cf. [12], see also Sect. 2.1). Let $\nabla^{H^\bullet(Z,F|_Z)} = \bigoplus_{i=0}^{\dim Z} \nabla^{H^i(Z,F|_Z)}$ denote the corresponding flat connection induced from ∇^F.

In [12], Bismut and Lott proved a Riemann–Roch–Grothendieck type formula for the imaginary part of the Cheeger–Chern–Simons character, which can be stated as an identity in $H^{odd}(B, \mathbb{R})$,

$$\int_Z e(TZ)\,\mathrm{Im}\left(\mathrm{CCS}\left(F, \nabla^F\right)\right) = \sum_{i=0}^{\dim Z} (-1)^i\,\mathrm{Im}\left(\mathrm{CCS}\left(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)}\right)\right). \tag{1.2}$$

They actually proved in [12] a refinement of (1.2) on the differential form level, and constructed a real analytic torsion form $\mathcal{T}(T^HM, g^{TZ}, h^F)$ (cf. (3.139), (3.137)) such that

$$\begin{aligned} d\mathcal{T}\left(T^HM, g^{TZ}, h^F\right) = {} & \int_Z e\left(TZ, \nabla^{TZ}\right) \sum_{j=0}^{+\infty} \frac{1}{j!} c_{2j+1}\left(F, h^F\right) \\ & - \sum_{j=0}^{+\infty} \frac{1}{j!} \sum_{i=0}^{\dim Z} (-1)^i c_{2j+1}\left(H^i(Z, F|_Z), h^{H^i(Z,F|_Z)}\right), \end{aligned} \tag{1.3}$$

where $e(TZ, \nabla^{TZ})$ and $c_{2j+1}(F, h^F)$ are defined in (3.84) and (2.42).

In this paper, we will present a new approach to (1.2) based on considerations of η-invariants of Atiyah–Patodi–Singer [2]. Besides giving a new proof of (1.2), our method also provides an extension of (1.2) to cover the real part of the Cheeger–Chern–Simons character.

One of the main results of this paper can be stated as the following identity in $H^{odd}(B, \mathbb{R}/\mathbb{Q})$,

$$\int_Z e(TZ)\,\mathrm{Re}\left(\mathrm{CCS}\left(F, \nabla^F\right)\right) = \sum_{i=0}^{\dim Z} (-1)^i\,\mathrm{Re}\left(\mathrm{CCS}\left(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)}\right)\right). \tag{1.4}$$

Putting (1.2) and (1.4) together, we get the following formula which can be thought of as a Riemann–Roch–Grothendieck formula for these Cheeger–Chern–Simons characters.

Theorem 1.1 *We have the following identity in $H^{odd}(B, \mathbf{C}/\mathbf{Q})$,*

$$\int_Z e(TZ)\,CCS\left(F, \nabla^F\right) = \sum_{i=0}^{\dim Z} (-1)^i\, CCS\left(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)}\right). \tag{1.5}$$

In particular, if $\mathbf{C}$ denotes the trivial complex line bundle over M, then one has

$$\sum_{i=0}^{\dim Z} (-1)^i\, CCS\left(H^i(Z, \mathbf{C}|_Z), \nabla^{H^i(Z,\mathbf{C}|_Z)}\right) = 0 \ \text{in} \ H^{odd}(B, \mathbf{C}/\mathbf{Q}). \tag{1.6}$$

It turns out that (1.4) has been obtained by Bismut in [9, Theorem 0.2] under the extra condition that TZ is fiber-wise oriented, while when $\dim Z$ is even (1.6) is a special case of [9, Theorem 3.12].

Our proof of (1.4), in its full generality, is based on an extension of [30, Theorem 0.2], where Zhang proved a Riemann–Roch type formula for certain extended versions of the Atiyah–Patodi–Singer η-invariant associated to the sub-signature operators constructed also in [30]. The main method used, as in [30], is the computation of the adiabatic limits of the constructed sub-signature operators, based on the techniques developed by Bismut–Cheeger [10] and Dai [20], as well as the local index computations developed in the papers of Bismut–Lott [12] and Bismut–Zhang [13]. Moreover, under suitable deformations of these sub-signature operators, the above arguments also lead to a new proof of (1.2). Thus, we obtain (1.5) solely in the framework of η-invariants.

It is particularly interesting that in such a process, the analytic torsion form of Bismut–Lott [12] shows up naturally in a transgression formula of the η-forms associated to the deformed operators. This suggests a natural relationship between the η and torsion invariants.

We should mention that the proof in [9] for (1.4) relies also on the computations of adiabatic limits of η-invariants. Moreover, when $\dim Z$ is even (1.6) plays a role in our proof of (1.4).

From another aspect, in view of the $\mathbf{R}/\mathbf{Z}$-index theory developed by Lott [25] (cf. Sect. 2.6), one can refine (1.4) to an identity in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$ if Z is even dimensional and spin^c (cf. Sect. 3.9): suppose that Z is even dimensional and spin^c. Let $S(TZ) = S^+(TZ) \oplus S^-(TZ)$ be the spinor bundle of TZ. In [25, Sect. 4], Lott defined a topological index Ind_{top} mapping from $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ to $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$. We denote by $\mathbf{C}$ the trivial complex line bundle carrying the trivial metric and connection. Let $\nabla^{F,e}$ be the Hermitian connection on (F, ∇^F, h^F) defined by (2.7). Then $\mathcal{F} = [(F, h^F, \nabla^{F,e}, 0) - \mathrm{rk}(F)\mathbf{C}] \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$. Set

$$I(F) = \sum_{i=0}^{\dim Z} (-1)^i \left(H^i(Z, F|_Z), h^{H^i(Z,F|_Z)}, \nabla^{H^i(Z,F|_Z),e}, 0\right). \tag{1.7}$$

Theorem 1.2 In $K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$, we have

$$\mathrm{Ind}_{top}\left(\left(S^{+}(TZ) - S^{-}(TZ)\right)\otimes \mathcal{F}\right) = I(F) - \mathrm{rk}(F)\, I(\mathbb{C}). \tag{1.8}$$

In fact, Bunke [16], Bunke and Schick [18] studied the index problem in their “smooth K-theory”, when the eta form appears naturally in their formalism. Bunke explained to us that Theorem 1.2 should hold without the assumption on the fiber Z, but one needs to use the twisted version of their theory.

On the other hand, for any integer $j \geq 0$, Cheeger and Simons defined in [19] (see also [9, (2.19), (3.4)]) the secondary character $c_{2j+1}(F, \nabla^F) \in H^{2j+1}(M, \mathbb{C}/\mathbb{Z})$. Similarly, one has the secondary characters $c_{2j+1}(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)})$, $0 \leq i \leq \dim Z$, in $H^{2j+1}(B, \mathbb{C}/\mathbb{Z})$.

Since $e(TZ)$ has integral periods, in view of Theorem 1.1 as well as its proof in Sect. 3, it is nature to formulate

Question 1.3 Whether the following identity holds for any integer $j \geq 0$ in $H^{2j+1}(B, \mathbb{C}/\mathbb{Z})$,

$$\begin{aligned}\int_Z e(TZ)\, c_{2j+1}\left(F, \nabla^F\right) &= \sum_{i=0}^{\dim Z} (-1)^i c_{2j+1}\left(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)}\right) \\ &\quad - \mathrm{rk}(F) \sum_{i=0}^{\dim Z} (-1)^i c_{2j+1}\left(H^i(Z, \mathbb{C}|_Z), \nabla^{H^i(Z,\mathbb{C}|_Z)}\right).\end{aligned} \tag{1.9}$$

There is also a topological proof of (1.2) given by Dwyer et al. [21]. It is an interesting question to know whether their method applies to (1.4).

On the other hand, Bloch and Esnault [14] (cf. the earlier works [22,23] and the survey [29]) defined certain algebraic Chern–Simons classes when E is an algebraic bundle over a smooth algebraic variety X (over an algebraic closed field k) admitting an algebraic flat connection ∇^E (Note that an algebraic bundle admits an algebraic connection if and only if its Atiyah class is zero), and developed certain Riemann–Roch type theorems in algebraic geometry (see [15] and the survey [24] for more details) which are closely related to the results of Bismut–Lott [12].

This paper is organized as follows. In Sect. 2, as in [12, Sect. 2], we deal with the finite dimensional situation. In Sect. 3, we develop a proof for both (1.2), (1.4), and (1.8), and discuss the relations between the η and torsion forms mentioned above.

2 η-invariants and flat cochain complexes

In this section, we discuss the η-invariants associated to a $\mathbb{Z}$-graded flat cochain complex. The framework is a combination of those in [10] and [12, Sect. 1, 2]. We show that the Bismut–Cheeger η-form is exact in computing the natural adiabatic limit of η-invariants appearing in the context. As a consequence, we deduce an equality relating the Cheeger–Chern–Simons characters of this cochain complex and of its cohomology. Moreover, a torsion form is constructed to transgress the -form. This torsion form turns out to be of the same nature as those constructed by Bismut–Lott in [12, Sect. 2].

This section is organized as follows. In Sect. 2.1, we set up the basic geometric data. In Sect. 2.2, we introduce a deformation for the twisted signature operator in the context. In Sect. 2.3, we compute the adiabatic limit of the -invariants associated to the deformed twisted signature operators discussed in Sect. 2.2. In Sect. 2.4, we recall the construction of the mod Q Cheeger–Chern–Simons character as well as its relation with -invariants. In Sect. 2.5, we establish a C/Q formula relating various Cheeger–Chern–Simons characters. In Sect. 2.6, we re ne the real part of the formula proved in Sect. 2.5 to an identity in the $K^{-1}_{R/Z}$-group. In Sect. 2.7, we construct the torsion form transgressing the -form mentioned above. In Sect. 2.8, we discuss in more detail the relationships between and torsion forms.

2.1 Superconnections and at cochain complexes

Let (E, v) be a Z-graded cochain complex of complex vector bundles over a compact smooth manifold B,

$$(E, v): \quad 0 \quad E^0 \ ^v \ E^1 \ ^v \cdots \ ^v \ E^n \quad 0. \tag{2.1}$$

Let $^E = \ _{i=0}^n \ ^{E^i}$ be a Z-graded connection on E. We call $(E, v, \ ^E)$ a at cochain complex if the following two identities hold,

$$^{E\ 2} = 0, \qquad ^E, v = 0, \tag{2.2}$$

where we have adopted the notation of supercommutator in the sense of Quillen [27].

Let $h^E = \ _{i=0}^n h^{E^i}$ be a Z-graded Hermitian metric on E.

Let $v \quad C \quad (B, \mathrm{Hom}(E \ , E^{-1}))$ be the adjoint of v with respect to h^E.

Let $(\ ^E)$ be the adjoint connection of ^{E} with respect to h^E.

By [13, (4.1),(4.2)] and [12, Sect. 1(g)], one has

$$^E \quad = \quad ^E + \quad E, h^E \tag{2.3}$$

with

$$E, h^E = \quad h^{E\ -1} \quad ^E h^E \ . \tag{2.4}$$

Let A , A be the superconnections (in the sense of Quillen [27]) on E de ned by

$$A = \ ^E + v, \qquad A = \quad ^E \quad + v \ . \tag{2.5}$$

Let $N \in \mathrm{End}(E)$ be the number operator of E, i.e., N acts on E^i by multiplication by i. We extend N to an element of $C^\infty(B, \mathrm{End}(E))$.

Following [12, (2.26), (2.30)], for any $u > 0$, set

$$C'_u = u^{N/2} A'' u^{-N/2} = \nabla^{E''} + \sqrt{u}v, \qquad C''_u = u^{-N/2} A' u^{N/2} = \nabla^{E'} + \sqrt{u}v^*, \qquad C_u = \frac{1}{2}\left(C'_u + C''_u\right), \quad D_u = \frac{1}{2}\left(C''_u - C'_u\right). \tag{2.6}$$

Let

$$\nabla^{E,e} = \nabla^{E} + \frac{1}{2}\omega\left(E, h^E\right) \tag{2.7}$$

be the Hermitian connection on (E, h^E) (cf. [12, (1.33)] and [13, (4.3)]). Then

$$C_u = \nabla^{E,e} + \frac{\sqrt{u}}{2}\left(v + v^*\right) \tag{2.8}$$

is a superconnection on E, while

$$D_u = \frac{1}{2}\omega\left(E, h^E\right) + \frac{\sqrt{u}}{2}\left(v^* - v\right) \tag{2.9}$$

is an element in $C^\infty(B, (\Lambda(T^*B) \otimes \mathrm{End}(E))^{\mathrm{odd}})$.

On the other hand, for any $b \in B$, let $H(E, v)_b = \bigoplus_{i=0}^{n} H^i(E, v)_b$ be the cohomology of the complex $(E, v)_b$. As in [12, Sect. 2(a)], by (2.2), there is a $\mathbb{Z}$-graded complex vector bundle $H(E, v)$ on B whose fiber over $b \in B$ is $H(E, v)_b$. Moreover, $H(E, v)$ carries a canonically induced flat connection $\nabla^{H(E,v)}$ (cf. [12, Proposition 2.5]). Indeed, let $P : \mathrm{Ker}(v) \to H(E, v)$ be the quotient map. Let s be a smooth section of $H(E, v)$, then locally, there is a smooth section s' of $\mathrm{Ker}(v)$ such that $P(s') = s$. By (2.2), $\nabla^E s' \in T^*B \otimes \mathrm{Ker}(v)$. Then $\nabla^{H(E,v)} s = P(\nabla^E s')$ defines the connection $\nabla^{H(E,v)}$.

Also, as in [12, Sect. 2(b)], it follows from the finite dimensional Hodge theory that for any $b \in B$, there is an isomorphism $H(E, v)_b \simeq \mathrm{Ker}((v + v^*)_b)$. Thus, there is a smooth $\mathbb{Z}$-graded sub-bundle $\mathrm{Ker}(v + v^*)$ of E whose fiber over $b \in B$ is $\mathrm{Ker}((v + v^*)_b)$, and

$$H(E, v) \simeq \mathrm{Ker}\left(v + v^*\right). \tag{2.10}$$

As a sub-bundle of E, $\mathrm{Ker}(v + v^*)$ inherits a Hermitian metric from the Hermitian metric h^E on E. Let $h^{H(E,v)}$ denote the Hermitian metric on $H(E, v)$ obtained via (2.10).

Let $p^{\mathrm{Ker}(v+v^*)}$ be the orthogonal projection from E onto $\mathrm{Ker}(v + v^*)$. It clearly preserves the $\mathbb{Z}$-grading.

By [12, Proposition 2.6], one knows that

$$\begin{aligned} P^{\mathrm{Ker}(v+v^*)}\nabla^{E}P^{\mathrm{Ker}(v+v^*)} &= \nabla^{H(E,v)},\\ P^{\mathrm{Ker}(v+v^*)}\omega\left(E,h^{E}\right)P^{\mathrm{Ker}(v+v^*)} &= \omega\left(H(E,v),h^{H(E,v)}\right),\\ P^{\mathrm{Ker}(v+v^*)}\nabla^{E,e}P^{\mathrm{Ker}(v+v^*)} &= \nabla^{H(E,v),e}. \end{aligned} \tag{2.11}$$

2.2 Twisted signature operators and their deformations

We assume in the rest of this section that $p = \dim B$ is odd and B is oriented.

Let g^{TB} be a Riemannian metric on TB.

For $X \in TB$, let $c(X)$, $\hat{c}(X)$ be the Clifford actions on $\Lambda(T^*B)$ defined by

$$c(X) = X^* - i_X, \quad \hat{c}(X) = X^* + i_X, \tag{2.12}$$

where $X^* \in T^*B$ corresponds to X via g^{TB} (cf. [12, (3.18)] and [13, Sect. 4(d)]). Then for any $X, Y \in TB$,

$$\begin{aligned} c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y\rangle,\\ \hat{c}(X)\hat{c}(Y) + \hat{c}(Y)\hat{c}(X) &= 2\langle X, Y\rangle,\\ c(X)\hat{c}(Y) + \hat{c}(Y)c(X) &= 0. \end{aligned} \tag{2.13}$$

Let $e_1, \ldots, e_p$ be a (local) oriented orthonormal basis of TB.

Let N_B be the number operator on $\Lambda(T^*B)$, i.e., N_B acts on $\Lambda^j(T^*B)$ by multiplication by j.

Set

$$\tau = \left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}}(-1)^{N_B+p}c(e_1)\cdots c(e_p) = \left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}}\hat{c}(e_1)\cdots\hat{c}(e_p). \tag{2.14}$$

Then τ is a well-defined self-adjoint element such that

$$\tau^2 = \mathrm{Id}|_{\Lambda(T^*B)}.$$

Let μ be a Hermitian vector bundle on B carrying a Hermitian connection ∇^μ with the curvature denoted by $R^\mu = \nabla^{\mu,2}$.

Let ∇^{TB} be the Levi-Civita connection on (TB, g^{TB}) with its curvature R^{TB}. Let $\nabla^{\Lambda(T^*B)}$ be the Hermitian connection on $\Lambda(T^*B)$ canonically induced from ∇^{TB}.

Let $\nabla^{\Lambda(T^*B)\otimes\mu\otimes E,e}$ be the tensor product connection on $\Lambda(T^*B)\otimes\mu\otimes E$ given by

$$\nabla^{\Lambda(T^*B)\otimes\mu\otimes E,e} = \nabla^{\Lambda(T^*B)}\otimes\mathrm{Id}_{\mu\otimes E} + \mathrm{Id}_{\Lambda(T^*B)}\otimes\nabla^{\mu}\otimes\mathrm{Id}_E + \mathrm{Id}_{\Lambda(T^*B)\otimes\mu}\otimes\nabla^{E,e}. \tag{2.15}$$

Let the Clifford actions c, $\hat{c}$ extend to actions on $\Lambda(T^*B)\otimes\mu\otimes E$ by acting as identity on $\mu\otimes E$.

Let be the induced $\mathbb{Z}_2$-grading operator on E, i.e., $= (Š1)^N$ on E. We extend to an action on $(T B)$ µ E by acting as identity on $(T B)$ µ.

DeÞnition 2.1 Let $D^{µ\ E}_{sig}$ be the (twisted) signature operator de ned by

$$D^{µ\ E}_{sig} = \sum_{i=1}^{p} c(e_i) \quad {}^{even}_{e_i}(T B)\ µ\ E, e : C \quad B, {}^{even}(T B)\ µ\ E$$
$$C \quad B, {}^{even}(T B)\ µ\ E \ . \tag{2.16}$$

One veri es that $D^{µ\ E}_{sig}$ is a formally self-adjoint rst order elliptic differential operator (cf. [5, Sect. 3.3]).

Let v, v extend to actions on ${}^{even}(T B)$ µ E by acting as identity on $(T B)$ µ.

For any u 0, set

$$D^{µ\ E}_{sig,u} = D^{µ\ E}_{sig} + \frac{\overline{u}}{2}(v + v). \tag{2.17}$$

Remark 2.2 $D^{µ\ E}_{sig,u}$ can be thought of as obtained from a (signature) quantization of C_u. Indeed, if B is spin, then one can consider the twisted Dirac operators instead of signature operators.

Let Y_u be the skew-adjoint element in $End({}^{even}(T B)\ µ\ E)$ de ned by

$$Y_u = \frac{}{2} \sum_{i=1}^{p} c(e_i) \quad E, h^E \ (e_i) + \frac{\overline{u}}{2}\ v\ Š\ v \ . \tag{2.18}$$

DeÞnition 2.3 For any r R and u 0, let $D^{µ\ E}_{sig,u}(r)$ be the operator de ned by

$$D^{µ\ E}_{sig,u}(r) = D^{µ\ E}_{sig,u} + \overline{Š1}\, r Y_u : C \ (B, {}^{even}(T B)\ µ\ E)$$
$$C \ (B, {}^{even}(T B)\ µ\ E). \tag{2.19}$$

Clearly, $D^{µ\ E}_{sig,u}(r)$ is still elliptic and formally self-adjoint.

For any X $T B$, set

$$c(X) = \quad c(X). \tag{2.20}$$

As p is odd, one veri es that for any X, Y $T B$,

$$c(X)c(Y) + c(Y)c(X) = Š\ 2\ X, Y \ . \tag{2.21}$$

From (2.16) to (2.20), one deduces that

$$D^{\mu\otimes E}_{sig,u}(r) = \sum_{i=1}^{p} c(e_i)\left(\nabla^{\Lambda^{even}(T^*B)\otimes\mu\otimes E,e}_{e_i} + \frac{\sqrt{-1}r}{2}\,\omega\left(E,h^E\right)(e_i)\right) + \frac{\sqrt{u}}{2}\left(\left(1-\sqrt{-1}r\right)v + \left(1+\sqrt{-1}r\right)v\right). \tag{2.22}$$

Remark 2.4 One verifies that $c(X)$, $X\in TB$, anti-commutes with elements in $\mathrm{End}^{odd}(E)$. Thus, we see that we are in a situation closely related to what considered in [10, Sect. 2(a)].

2.3 A computation of adiabatic limits of η-invariants

Let $\bar\eta(D^{\mu\otimes E}_{sig,u}(r))$ be the reduced η-invariant in the sense of Atiyah–Patodi–Singer [2]. More precisely, for $s\in\mathbb{C}$, $\mathrm{Re}(s) > p$, set

$$\eta\left(D^{\mu\otimes E}_{sig,u}(r)\right)(s) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^{+\infty} t^{\frac{s-1}{2}}\,\mathrm{Tr}\left[D^{\mu\otimes E}_{sig,u}(r)\exp\left(-tD^{\mu\otimes E}_{sig,u}(r)^2\right)\right]dt. \tag{2.23}$$

Then $\eta(D^{\mu\otimes E}_{sig,u}(r))(s)$ extends to a meromorphic functions of $s\in\mathbb{C}$ and is holomorphic at $s = 0$.

The reduced η-invariant of $D^{\mu\otimes E}_{sig,u}(r)$ is defined by

$$\bar\eta\left(D^{\mu\otimes E}_{sig,u}(r)\right) = \frac{1}{2}\left(\eta\left(D^{\mu\otimes E}_{sig,u}(r)\right)(0) + \dim \mathrm{Ker}\left(D^{\mu\otimes E}_{sig,u}(r)\right)\right). \tag{2.24}$$

By [10, Theorem 2.7], one knows that for any $u\geq 0$,

$$\bar\eta\left(D^{\mu\otimes E}_{sig,u}(r)\right) \equiv \bar\eta\left(D^{\mu\otimes E}_{sig}(r)\right) \mod \mathbb{Z}, \tag{2.25}$$

where $D^{\mu\otimes E}_{sig}(r)$ is the notation for $D^{\mu\otimes E}_{sig,u=0}(r)$ for brevity.

We fix a square root of $\sqrt{-1}$ and let $\varphi : \Lambda(T^*B)\to\Lambda(T^*B)$ be the homomorphism defined by $\varphi : \omega\in\Lambda^i(T^*B)\to\left(2\pi\sqrt{-1}\right)^{-i/2}\omega$. The formulas in what follows will not depend on the choice of the square root of $\sqrt{-1}$.

Let η_r be the η-form of Bismut–Cheeger [10, (2.26)] defined by

$$\eta_r = \frac{1}{\left(2\pi\sqrt{-1}\right)^{\frac{1}{2}}}\,\varphi\int_0^{+\infty}\mathrm{Tr}_s\left[\left(\left(1-\sqrt{-1}r\right)v + \left(1+\sqrt{-1}r\right)v\right)e^{-\left(C_u+\sqrt{-1}rD_u\right)^2}\right]\frac{du}{4\sqrt{u}}, \tag{2.26}$$

where Tr_s is the supertrace on E in the sense of Quillen [27] with respect to the $\mathbb{Z}_2$-grading induced by $(-1)^N$, i.e., $\mathrm{Tr}_s[\cdot] = \mathrm{Tr}[(-1)^N \cdot]$.

Remark 2.5 Since $\mathrm{Ker}(v + v^*)$ forms a vector bundle over B and

$$\left((1-\sqrt{-1}r)v + (1+\sqrt{-1}r)v^*\right)^2 = (1+r^2)\left(v+v^*\right)^2,$$

by [10, Lemma 2.1] and [5, Sect. 9.1], η_r in (2.26) is well-defined.

Now for any $r \in \mathbb{R}$, let $D^{\mu\otimes H(E,v)}_{\mathrm{sig}}(r)$ be the deformed twisted signature operator defined by replacing (E, v, ∇^E, h^E) by $(H(E,v), 0, \nabla^{H(E,v)}, h^{H(E,v)})$. That is, let $\nabla^{\Lambda(T^*B)\otimes\mu\otimes H(E,v),e}$ be the connection on $\Lambda(T^*B)\otimes\mu\otimes H(E,v)$ induced by $\nabla^{\Lambda(T^*B)}$, ∇^{μ} and $\nabla^{H(E,v),e}$, then

$$D^{\mu\otimes H(E,v)}_{\mathrm{sig}}(r) = \sum_{i=1}^{p} c(e_i)\left(\nabla_{e_i}^{\Lambda^{\mathrm{even}}(T^*B)\otimes\mu\otimes H(E,v),e} + \frac{\sqrt{-1}r}{2}\,\omega\left(H(E,v), h^{H(E,v)}\right)(e_i)\right). \tag{2.27}$$

Theorem 2.6 For any $r \in \mathbb{R}$, the following identity holds,

$$\bar\eta\left(D^{\mu\otimes E}_{\mathrm{sig}}(r)\right) \equiv \bar\eta\left(D^{\mu\otimes H(E,v)}_{\mathrm{sig}}(r)\right) \mod \mathbb{Z}. \tag{2.28}$$

Proof By (2.25), (2.28) is equivalent to

$$\lim_{u\to+\infty} \bar\eta\left(D^{\mu\otimes E}_{\mathrm{sig},u}(r)\right) \equiv \bar\eta\left(D^{\mu\otimes H(E,v)}_{\mathrm{sig}}(r)\right) \mod \mathbb{Z}. \tag{2.29}$$

Now by Remark 2.5 and by proceeding as in [10, Theorem 2.28], one knows that when $v + v^*$ is invertible, i.e., when $H(E,v) = \{0\}$, one has

$$\lim_{u\to+\infty} \bar\eta\left(D^{\mu\otimes E}_{\mathrm{sig},u}(r)\right) \equiv \int_B L\left(TB, \nabla^{TB}\right) \mathrm{ch}\left(\mu, \nabla^{\mu}\right) \eta_r, \tag{2.30}$$

where $L(TB, \nabla^{TB})$ is the Hirzebruch characteristic form defined by

$$L(TB, \nabla^{TB}) = \det{}^{1/2}\left(\frac{R^{TB}}{\tanh\left(R^{TB}/2\right)}\right),$$

while $\mathrm{ch}(\mu, \nabla^{\mu})$ is the Chern character form defined by

$$\mathrm{ch}(\mu, \nabla^{\mu}) = \mathrm{Tr}\left[\exp\left(-R^{\mu}\right)\right].$$

While in the general case where Ker$(v + v)$ forms a vector bundle over B, one can generalize the arguments in [10, Theorem 2.28] to show that when mod Z,

$$\lim_{u \to +} - D^{\mu}_{\mathrm{sig},u}{}^{E}(r) \quad - D^{\mu}_{\mathrm{sig}}{}^{H(E,v)}(r) \; + \; \int_B L \quad TB, \quad {}^{TB} \; \mathrm{ch} \; \mu, \quad {}^{\mu} \quad {}_r . \tag{2.31}$$

Remark 2.7 Indeed, see [8, Theorem 2.39] for a very simple proof of (2.31).

Lemma 2.8 For any $r \in \mathbb{R}$, $_r$ is exact. Moreover,

$$_{r=0} = 0. \tag{2.32}$$

Proof From (2.6), one verifies directly that for any $r \in \mathbb{R}$,

$$\left(C_u + \sqrt{-1} r D_u\right)^2 = \left(1 + r^2\right) C_u^2 = -\left(1 + r^2\right) D_u^2. \tag{2.33}$$

By (2.8), (2.9),

$$\frac{1}{2\sqrt{u}} \quad v + v \quad = \frac{-1}{u} [N, D_u] \, , \quad \frac{1}{2\sqrt{u}} \quad v - v \quad = \frac{-1}{u} [N, C_u] \, , \tag{2.34}$$

from which one gets that for any $r \in \mathbb{R}$ and $u > 0$,

$$\begin{aligned} &\frac{1}{4\sqrt{u}} \mathrm{Tr}_s \quad 1 - \sqrt{-1} r \quad v + \quad 1 + \sqrt{-1} r \quad v \quad e^{-(C_u + \sqrt{-1} r D_u)^2} \\ &= \frac{-1}{2u} \mathrm{Tr}_s \quad [N, D_u] + \sqrt{-1} r \, [N, C_u] \quad e^{- \; C_u + \sqrt{-1} r D_u \; ^2} \\ &= \frac{\sqrt{-1} r}{2u} d \mathrm{Tr}_s \quad N e^{- \; C_u + \sqrt{-1} r D_u \; ^2} \quad . \end{aligned} \tag{2.35}$$

From (2.26) and (2.35), one sees that $_r$ is an exact form. In particular, by setting $r = 0$ in (2.35) and by (2.26), one gets (2.32).

Combining Lemma 2.8 with (2.31), one gets (2.29), which completes the proof of Theorem 2.6.

Remark 2.9 The transgression formula (2.35) suggests that it is possible to transgress the -form $_r$ (2.26) through torsion like forms of the same nature as those of Bismut–Lott [12, De nition 3.22]. This will be dealt with in more details in Sect. 2.7.

2.4 Cheeger–Chern–Simons characters and -invariants

We rst recall the de nition of the mod Q Cheeger–Chern–Simons character for at vector bundles.

Let W be a complex vector bundle over B. Let ∇_0^W, ∇_1^W be two connections on W. Let ∇_t^W, $0 \le t \le 1$, be a smooth path of connections on W connecting ∇_0^W and ∇_1^W.

Let $CS(\nabla_0^W, \nabla_1^W)$ be the differential form defined by

$$CS\left(\nabla_0^W, \nabla_1^W\right) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{1}{2}} \int_0^1 \mathrm{Tr}\left[\frac{\partial \nabla_t^W}{\partial t} \exp\left(-\left(\nabla_t^W\right)^2\right)\right] dt. \tag{2.36}$$

Then

$$dCS\left(\nabla_0^W, \nabla_1^W\right) = \mathrm{ch}\left(W, \nabla_1^W\right) - \mathrm{ch}\left(W, \nabla_0^W\right). \tag{2.37}$$

Moreover, up to exact forms, $CS(\nabla_0^W, \nabla_1^W)$ does not depend on the path of connections on W connecting ∇_0^W and ∇_1^W.

Let (E^i, ∇^{E^i}), $i = 1, 2, 3$, be complex vector bundles with connections such that there is a short exact sequence

$$0 \to E^0 \xrightarrow{i} E^1 \xrightarrow{} E^2 \to 0.$$

Take a splitting map $f : E^2 \to E^1$. Then $i + f : E^0 \oplus E^2 \to E^1$ is an isomorphism.

The Chern–Simons class $CS(\nabla^{E^0}, \nabla^{E^1}, \nabla^{E^2}) \in \Omega^{odd}(M)/\mathrm{Im}(d)$ is defined by (cf. [25, (10)])

$$CS\left(\nabla^{E^0}, \nabla^{E^1}, \nabla^{E^2}\right) := CS\left((i + f)^* \nabla^{E^1}, \nabla^{E^0} \oplus \nabla^{E^2}\right). \tag{2.38}$$

The class $CS(\nabla^{E^0}, \nabla^{E^1}, \nabla^{E^2})$ does not depend on the choice of the splitting map f.

Remark 2.10 If $0 \to E^0 \to E^1 \to E^2 \to 0$ is a short exact sequence of flat vector bundles as in (2.1). By Lemma 2.8 and the characterization of the form (cf. [8, Theorem 2.10] and [17, Lemma 3.16]), we know that in $\Omega^{odd}(M)/\mathrm{Im}(d)$,

$$CS\left(\nabla^{E^0,e}, \nabla^{E^1,e}, \nabla^{E^2,e}\right) = \big|_{r=0} = 0. \tag{2.39}$$

Now let (F, ∇^F) be a complex flat vector bundle over B. Then by [1], there is a positive integer q such that qF, the direct sum of q copies of F, is topologically trivial. Let ∇_0^{qF} be a trivial connection on qF which can be determined by choosing a global basis of qF. Let $q\nabla^F$ be the natural flat connection on qF obtained from the direct sum of q copies of ∇^F.

By (2.37), one sees that $CS(\nabla_0^{qF}, q\nabla^F)$ is a closed form on B. Moreover, by proceeding as in [25, Lemma 1], one shows that $\frac{1}{q} CS(\nabla_0^{qF}, q\nabla^F)$ determines a cohomology class in $H^{odd}(B, \mathbf{C}/\mathbf{Q})$ not depending on the choice of q and ∇_0^{qF}.

Definition 2.11 We define the mod $\mathbf{Q}$ Cheeger–Chern–Simons character of (F, ∇^F) to be

$$CCS(F, \nabla^F) = \frac{1}{q} CS\left(\nabla_0^{qF}, q\nabla^F\right) \in H^{odd}(B, \mathbf{C}/\mathbf{Q}). \tag{2.40}$$

By [19, Proposition 2.9], up to $\mathrm{rk}(F)$, $CCS(F, \nabla^F)$ is exactly $ch(F, \nabla^F) \in H^{odd}(B, \mathbf{C}/\mathbf{Q})$ defined in [9, (2.19), Theorem 2.3].

Let h^F be a Hermitian metric on F. Let $\omega(F, h^F)$ be given similarly as in (2.4), and let $\nabla^{F,e}$ be the associated Hermitian connection on F given similarly as in (2.7). Let $q\nabla^{F,e}$ be the connection on qF obtained from the direct sum of q copies of $\nabla^{F,e}$.

One verifies directly that

$$\begin{aligned} CCS\left(F, \nabla^F\right) &= \frac{1}{q} CS\left(\nabla_0^{qF}, q\nabla^{F,e}\right) + \frac{1}{q} CS\left(q\nabla^{F,e}, q\nabla^F\right) \\ &= \frac{1}{q} CS\left(\nabla_0^{qF}, q\nabla^{F,e}\right) + CS\left(\nabla^{F,e}, \nabla^F\right). \end{aligned} \tag{2.41}$$

Following [12, (0.2)], for any integer $j \geq 0$, let $c_{2j+1}(F, h^F)$ be the Chern form defined by

$$c_{2j+1}(F, h^F) = \left(2\sqrt{-1}\right)^{-j} 2^{-(2j+1)} \mathrm{Tr}\left[\omega^{2j+1}\left(F, h^F\right)\right]. \tag{2.42}$$

Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(B, \mathbf{R})$, which does not depend on the choice of h^F.

The following identity has been proved in [12, Proposition 1.14],

$$\frac{1}{\sqrt{-1}} CS\left(\nabla^{F,e}, \nabla^F\right) = \mathrm{Im}\left(CCS\left(F, \nabla^F\right)\right) = -\frac{1}{2} \sum_{j=0}^{+\infty} \frac{2^{2j}\, j!}{(2j+1)!} c_{2j+1}(F). \tag{2.43}$$

Consequently,

$$\mathrm{Re}\left(CCS\left(F, \nabla^F\right)\right) = \frac{1}{q} CS\left(\nabla_0^{qF}, q\nabla^{F,e}\right) \text{ in } H^{odd}(B, \mathbf{R}/\mathbf{Q}). \tag{2.44}$$

We now come to consider the η invariants mentioned in the title of this subsection.

Recall that B is compact, oriented, carrying a Riemannian metric g^{TB} and that $p = \dim B$ is odd. Recall also that μ is a Hermitian vector bundle over B carrying a Hermitian connection ∇^μ.

We apply the constructions in Sects. 2.1–2.3 to the trivial flat cochain complex $(F, 0, \nabla^F)$. Thus, let $D_{sig}^{\mu \otimes F}$ be the twisted signature operator defined as in (2.16).

Let h_0^{qF} be a Hermitian metric on qF such that ∇_0^{qF} is a Hermitian connection with respect to h_0^{qF}.

It is easy to see that one can construct a smooth pass of Hermitian metrics connecting q^*h^F and $h_0^{q^*F}$, as well as a smooth pass of Hermitian connections connecting $q^*\nabla^{F,e}$ and $\nabla_0^{q^*F}$.

By the standard variation formula for reduced η-invariants (cf. [2] and [11, Theorem 2.10]), one finds

$$\begin{aligned}\bar\eta\left(q^*D^{\mu}_{\mathrm{sig}}\otimes F\right)-q\,\mathrm{rk}(F)\,\bar\eta\left(D^{\mu}_{\mathrm{sig}}\right)\\ \equiv \int_B L\left(TB,\nabla^{TB}\right)\mathrm{ch}\left(\mu,\nabla^{\mu}\right)CS\left(\nabla_0^{q^*F},q^*\nabla^{F,e}\right)\ \bmod \mathbf{Z}.\end{aligned}\tag{2.45}$$

From (2.44) and (2.45), one gets

$$\bar\eta\left(D^{\mu}_{\mathrm{sig}}\otimes F\right)-\mathrm{rk}(F)\,\bar\eta\left(D^{\mu}_{\mathrm{sig}}\right)\equiv\int_B L(TB)\,\mathrm{ch}(\mu)\,\mathrm{Re}\left(CCS\left(F,\nabla^F\right)\right)\ \bmod \mathbf{Q}.\tag{2.46}$$

For any $r\in\mathbf{R}$, let $\nabla^{F,e,(r)}$ denote the Hermitian connection on F defined by

$$\nabla^{F,e,(r)}=\nabla^{F,e}+\frac{\sqrt{-1}\,r}{2}\,\omega\left(F,h^F\right).\tag{2.47}$$

For any integer $j\geq 0$ and $r\in\mathbf{R}$, let $a_j(r)\in\mathbf{R}$ be defined as

$$a_j(r)=\int_0^1\left(1+u^2r^2\right)^j du.\tag{2.48}$$

Lemma 2.12 *The following identity in $H^{\mathrm{odd}}(B,\mathbf{R})$ holds,*

$$CS\left(\nabla^{F,e},\nabla^{F,e,(r)}\right)=-\frac{r}{2}\sum_{j=0}^{+\infty}\frac{a_j(r)}{j!}c_{2j+1}(F).\tag{2.49}$$

Proof Formula (2.49) follows from (2.36), (2.42) and a direct computation in considering the smooth pass of connections $(1-u)\nabla^{F,e}+u\nabla^{F,e,(r)}$, $0\leq u\leq 1$.

Remark 2.13 By comparing (2.43) and (2.49), we see that up to rescaling, one can recover the imaginary part of the Cheeger–Chern–Simons character $CCS(F,\nabla^F)$ through (deformed) Hermitian connections.

From (2.22), (2.25), (2.49) and the standard variation formula for reduced η-invariants, one finds that for any $r \in \mathbb{R}$,

$$\bar{\eta}\left(D_{sig}^{\mu,F}(r)\right) - \bar{\eta}\left(D_{sig}^{\mu,F}\right) - \int_B L\left(TB, \nabla^{TB}\right) \mathrm{ch}\left(\mu, \nabla^{\mu}\right) CS\left(\nabla^{F,e}, \nabla^{F,e,(r)}\right) \mod \mathbb{Z}$$
$$= -\frac{r}{2}\int_B L(TB)\mathrm{ch}(\mu) \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} c_{2j+1}(F). \qquad (2.50)$$

2.5 Flat cochain complex and the Cheeger–Chern–Simons character

We make the same assumptions and use the same notation as in Sects. 2.1–2.3. Thus, (E, v, ∇^E) is a $\mathbb{Z}$-graded flat cochain complex over B, etc.

Let $CCS(E, \nabla^E)$ denote the $\mathbb{C}/\mathbb{Q}$ Cheeger–Chern–Simons character defined by

$$CCS\left(E, \nabla^E\right) = \sum_{i=0}^{n} (-1)^i CCS\left(E^i, \nabla^{E^i}\right) \quad \text{in } H^{odd}(B, \mathbb{C}/\mathbb{Q}). \qquad (2.51)$$

The imaginary part of the following result has been proved by Bismut–Lott [12, Theorem 2.19].

Theorem 2.14 The following identity holds in $H^{odd}(B, \mathbb{C}/\mathbb{Q})$,

$$CCS\left(E, \nabla^E\right) = CCS\left(H(E,v), \nabla^{H(E,v)}\right). \qquad (2.52)$$

Proof By Theorem 2.6, we know that for any $r \in \mathbb{R}$,

$$\bar{\eta}\left(D_{sig}^{\mu,E}(r)\right) - \bar{\eta}\left(D_{sig}^{\mu,E}\right) \equiv \bar{\eta}\left(D_{sig}^{\mu,H(E,v)}(r)\right) - \bar{\eta}\left(D_{sig}^{\mu,H(E,v)}\right) \mod \mathbb{Z}. \qquad (2.53)$$

From (2.50) and (2.53), one finds

$$\frac{r}{2}\int_B L(TB)\mathrm{ch}(\mu) \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} \sum_{i=0}^{n} (-1)^i c_{2j+1}\left(E^i\right)$$
$$= \frac{r}{2}\int_B L(TB)\mathrm{ch}(\mu) \sum_{j=0}^{+\infty} \frac{a_j(r)}{j!} \sum_{i=0}^{n} (-1)^i c_{2j+1}\left(H^i(E,v)\right) \mod \mathbb{Z}. \qquad (2.54)$$

By taking derivative with respect to r at $r=0$, one gets that

$$\int_B L(TB)\operatorname{ch}(\mu)\sum_{j=0}^{+\infty}\frac{1}{j!}\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(E^i,\nabla^{E^i}\right)$$
$$=\int_B L(TB)\operatorname{ch}(\mu)\sum_{j=0}^{+\infty}\frac{1}{j!}\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(H^i(E,v),\nabla^{H^i(E,v)}\right). \tag{2.55}$$

Since (2.55) holds for any complex vector bundle μ over B, while $L(TB)\operatorname{ch}(\cdot): K(B)\otimes\mathbb{Q}\to H^{\mathrm{even}}(B,\mathbb{Q})$ is an isomorphism (cf. [1]), from (2.55) and a simple degree counting, one deduces that for any integer $j\geq 0$,

$$\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(E^i,\nabla^{E^i}\right)=\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(H^i(E,v),\nabla^{H^i(E,v)}\right)\quad\text{in } H^{2j+1}(B,\mathbb{R}). \tag{2.56}$$

From (2.43), (2.51) and (2.56), one gets

$$\operatorname{Im}\left(\mathrm{CCS}\left(E,\nabla^E\right)\right)=\operatorname{Im}\left(\mathrm{CCS}\left(H(E,v),\nabla^{H(E,v)}\right)\right), \tag{2.57}$$

which was first proved in [12, Theorem 2.19] by using a direct transgression method.

Now by applying (2.46) to each E^i as well as each $H^i(E,v)$, $0\leq i\leq n$, and by Theorem 2.6, one finds that

$$\int_B L(TB)\operatorname{ch}(\mu)\operatorname{Re}\left(\mathrm{CCS}\left(E,\nabla^E\right)\right)$$
$$\equiv\int_B L(TB)\operatorname{ch}(\mu)\operatorname{Re}\left(\mathrm{CCS}\left(H(E,v),\nabla^{H(E,v)}\right)\right)\mod\mathbb{Q}. \tag{2.58}$$

By using the fact that $L(TB)\operatorname{ch}(\cdot): K(B)\otimes\mathbb{Q}\to H^{\mathrm{even}}(B,\mathbb{Q})$ is an isomorphism again, one deduces from (2.58) the following identity in $H^{\mathrm{odd}}(B,\mathbb{R}/\mathbb{Q})$,

$$\operatorname{Re}\left(\mathrm{CCS}\left(E,\nabla^E\right)\right)=\operatorname{Re}\left(\mathrm{CCS}\left(H(E,v),\nabla^{H(E,v)}\right)\right). \tag{2.59}$$

From (2.57) and (2.59), one gets (2.52).

2.6 A refinement in $K^{-1}_{\mathbb{R}/\mathbb{Z}}(B)$

In the discussions in the previous subsections, we have only assumed that B is oriented, and this is why we have used the twisted signature operators. If B is spin^c or even spin, then we can well use the twisted Dirac operators instead. In particular, this will enable us to apply the constructions in the $\mathbb{R}/\mathbb{Z}$-index theory developed by Lott [25] to the current situation, where the form (at $r=0$) vanishes tautologically.

We form the abelian semigroup $K^{-1}_{R/Z}(B)$ consisting of isomorphism classes of tuples (E, h^E, ∇^E, η), where $E = E_+ \oplus E_-$ is a Z_2-graded complex vector bundle, h^E is a Hermitian metric and $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a metric connection, both being compatible with the grading, and $\eta \in \Omega^{odd}(B)/\operatorname{Im}(d)$ satisfies

$$d\eta = \operatorname{ch}\left(E, \nabla^E\right) := \operatorname{ch}\left(E_+, \nabla^{E_+}\right) - \operatorname{ch}\left(E_-, \nabla^{E_-}\right).$$

The semigroup operation is induced by direct sums of generators.

On $K^{-1}_{R/Z}(B)$ we consider the minimal equivalence relation $\sim$ which is compatible with the semigroup structure and such that the following properties hold,

(1) change of connections: We have $(E, h^E, \nabla, \eta) \sim (E, h^E, \nabla', \eta')$ iff $\eta' = \eta + CS(\nabla, \nabla')$,

(2) trivial elements: If (E, h^E, ∇^E) is a Z_2-graded Hermitian vector bundle with connection, then $(E \oplus E^{op}, h^E \oplus h^E, \nabla^E \oplus \nabla^E, 0) \sim 0$, where E^{op} denotes E with the opposite grading.

The group $K^{-1}_{R/Z}(B)$ is the quotient of $K^{-1}_{R/Z}(B)$ by $\sim$. We still use (E, h^E, ∇^E, η) to denote the class of (E, h^E, ∇^E, η) in $K^{-1}_{R/Z}(B)$.

It was shown by Lott [25, Sect. 2] that the group $K^{-1}_{R/Z}(B)$ given by this geometric definition is naturally isomorphic to the topological definition.

Let (E, ∇^E) be a flat vector bundle on B carrying a Hermitian metric h^E as in (2.1), one easily sees that

$$\left(E, h^E, \nabla^{E,e}, 0\right) = \sum_{i=0}^{n} (-1)^i \left(E^i, h^{E^i}, \nabla^{E^i,e}, 0\right)$$

is an element in $K^{-1}_{R/Z}(B)$.

Theorem 2.15 The following identity holds in $K^{-1}_{R/Z}(B)$,

$$\left(E, h^E, \nabla^{E,e}, 0\right) = \left(H(E,v), h^{H(E,v)}, \nabla^{H(E,v),e}, 0\right). \tag{2.60}$$

Proof Clearly, (2.60) is a refinement of the real part of (2.52). It is also a direct consequence of [25, Definition 6] and Remark 2.10. In fact, let $F^i = \operatorname{Im}(v_{E^{i-1}})$, $G^i = \operatorname{Ker}(v_{E^i})$, then F^i, G^i are flat vector bundles on B with Hermitian metrics induced by h^E. Now we have short exact sequences of flat vector bundles: $0 \to F^i \to G^i \to H^i(E,v) \to 0$, $0 \to G^i \to E^i \to F^{i+1} \to 0$. Then by [25, Definition 6] and Remark 2.10, in $K^{-1}_{R/Z}(B)$,

$$\left(G^i, \nabla^{G^i,e}, 0\right) = \left(F^i, \nabla^{F^i,e}, 0\right) + \left(H^i(E,v), \nabla^{H^i(E,v),e}, 0\right),$$

$$\left(E^i, \nabla^{E^i,e}, 0\right) = \left(G^i, \nabla^{G^i,e}, 0\right) + \left(F^{i+1}, \nabla^{G^{i+1},e}, 0\right).$$

Thus we get (2.60).

2.7 Torsion forms and a transgression formula for $_r$

As in [12, (2.39)], we denote

$$d(E) = \sum_{i=0}^{n} (-1)^i i\, \mathrm{rk}\, E^i \;, \quad d(H(E,v)) = \sum_{i=0}^{n} (-1)^i i\, \mathrm{rk}\, H^i(E,v) \;. \tag{2.61}$$

By (2.33) and by [12, Theorem 2.13 and Proposition 2.18], one has that as $u \; +$,

$$\mathrm{Tr}_s\left[N e^{-\left(C_u + \sqrt{-1} r D_u\right)^2}\right] = d(H(E,v)) + O\left(\frac{1}{\sqrt{u}}\right), \tag{2.62}$$

and that when $u \quad 0^+$,

$$\mathrm{Tr}_s\left[N e^{-\left(C_u + \sqrt{-1} r D_u\right)^2}\right] = d(E) + O(u). \tag{2.63}$$

The following definition is closely related to [12, Definition 2.20].

Definition 2.16 For any $r \quad R$, put

$$I_r = - r \int_0^{+} \Big\{ \mathrm{Tr}_s\left[N e^{-(C_u + \sqrt{-1} r D_u)^2}\right] - d(H(E,v)) - (d(E) - d(H(E,v)))\, e^{-u/4} \Big\} \frac{du}{2u}. \tag{2.64}$$

Theorem 2.17 For any $r \quad R$, the following transgression formula holds,

$$_r = -\frac{1}{2} d I_r . \tag{2.65}$$

Proof Formula (2.65) follows from (2.26), (2.35), (2.62)–(2.64).

Let $T_f(A, h^E)$ be the torsion form constructed in [12, Definition 2.20] associated to the odd holomorphic function $f(z)$ such that $f\ (z) = e^{z^2}$, that is,

$$T_f\left(A, h^E\right) = - \int_0^{+} \Big\{ \mathrm{Tr}_s\left[N e^{D_u^2}\right] - d(H(E,v)) - (d(E) - d(H(E,v)))\, e^{-u/4} \Big\} \frac{du}{2u}. \tag{2.66}$$

Theorem 2.18 The following identity holds,

$$\frac{I_r}{r}\Big|_{r=0} = T_f\left(A, h^E\right). \tag{2.67}$$

In particular,

$$\frac{r}{r}\Big|_{r=0} = Š\ \frac{1}{2} d T_f \quad A, h^E \quad . \tag{2.68}$$

Proof Formula (2.67) follows from (2.64) and [12, De nition 2.20]. Formula (2.68) follows from (2.65) and (2.67).

Combining (2.64), (2.65) with the Bismut–Lott transgression formula [12, Theorem 2.22] for the function $h(x) = \sum_{j=0}^{+} \frac{(1+r^2)^j}{j!} \frac{x^{2j+1}}{2j+1}$, one gets

Corollary 2.19 For any r R, the following identity holds,

$$\begin{aligned} {}_r = \frac{r}{2} \sum_{j=0}^{+} \frac{(1+r^2)^j}{j!} \sum_{i=0}^{n} \frac{(Š1)^i}{2j+1} c_{2j+1} \quad H^i(E,v), h^{H^i(E,v)} \\ Š \frac{r}{2} \sum_{j=0}^{+} \frac{(1+r^2)^j}{j!} \sum_{i=0}^{n} \frac{(Š1)^i}{2j+1} c_{2j+1} \quad E^i, h^{E^i} \quad . \end{aligned} \tag{2.69}$$

In particular,

$$\begin{aligned} \frac{r}{r}\Big|_{r=0} = \frac{1}{2} \sum_{j=0}^{+} \frac{1}{j!} \sum_{i=0}^{n} \frac{(Š1)^i}{2j+1} c_{2j+1} \quad H^i(E,v), h^{H^i(E,v)} \\ Š \frac{1}{2} \sum_{j=0}^{+} \frac{1}{j!} \sum_{i=0}^{n} \frac{(Š1)^i}{2j+1} c_{2j+1} \quad E^i, h^{E^i} \quad . \end{aligned} \tag{2.70}$$

Remark 2.20 In view of Theorems 2.17 and 2.18, a direct computation of (2.69) or (2.70) will lead to an alternate proof of the Bismut–Lott transgression formula [12, Theorem 2.22].

2.8 More on and torsion forms

On $B \times R \times R_+$, let $C + \overline{Š1} r D$ be the operator de ned by

$$C + \overline{Š1} r D \Big|_{B\times\{r\}\times\{u\}} = C_u + \overline{Š1} r D_u + dr \frac{}{r} + du \frac{}{u}. \tag{2.71}$$

Then $\mathrm{Tr}_s \ \exp\ Š(C + \overline{Š1} r D)^2$ is closed on $B \times R \times R_+$, moreover

$$C + \overline{Š1} r D^{\ 2} = C_u + \overline{Š1} r D_u^{\ 2} + du \frac{}{u} \ C_u + \overline{Š1} r D_u + \overline{Š1} dr D_u. \tag{2.72}$$

By the Volterra expansion formula [5, Sect. 2.4], we get

$$
\begin{aligned}
&\mathrm{Tr}_s\left[\exp\left(-\left(C+\sqrt{-1}rD\right)^2\right)\right]\\
&= \mathrm{Tr}_s\left[\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&\quad - du\,\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}rD_u\right)\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&\quad - dr\,\mathrm{Tr}_s\left[\sqrt{-1}D_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&\quad + \int_0^1 ds\,\mathrm{Tr}_s\left[du\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}rD_u\right)\exp\left(-s\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right.\\
&\quad \left.\times\sqrt{-1}dr\,D_u\exp\left(-(1-s)\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right].
\end{aligned} \tag{2.73}
$$

Applying the total differentiation $d^{B\times\mathbb{R}\times\mathbb{R}_+}$ on $B\times\mathbb{R}\times\mathbb{R}_+$ to (2.73), after comparing the coefficients of $du\,dr$, and using the fact that D_u commutes with $\exp\left(-s(C_u+\sqrt{-1}rD_u)^2\right)$, we get

$$
\begin{aligned}
&\frac{\partial}{\partial r}\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}rD_u\right)\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&\quad -\frac{\partial}{\partial u}\mathrm{Tr}_s\left[\sqrt{-1}D_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&= d\,\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}rD_u\right)\sqrt{-1}D_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right].
\end{aligned} \tag{2.74}
$$

Now by (2.34),

$$
\begin{aligned}
&\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}rD_u\right)\sqrt{-1}D_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&= -\frac{\sqrt{-1}}{2u}\mathrm{Tr}_s\left[\left([N_Z,D_u]+\left[N_Z,\sqrt{-1}rC_u\right]\right)D_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&= -\frac{\sqrt{-1}}{2u}\mathrm{Tr}_s\left[2N_Z D_u^2\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right]\\
&\quad -\frac{1}{2u}d\,\mathrm{Tr}_s\left[N_Z rD_u\exp\left(-\left(C_u+\sqrt{-1}rD_u\right)^2\right)\right].
\end{aligned} \tag{2.75}
$$

From (2.35) and (2.74)–(2.75), one deduces that

$$\frac{\partial}{\partial r} d\mathrm{Tr}_s\left[\frac{rN}{2u} e^{-(1+r^2)C_u^2}\right] - \frac{\partial}{\partial u}\mathrm{Tr}_s\left[D_u e^{-(1+r^2)C_u^2}\right] = -\frac{1}{u} d\mathrm{Tr}_s\left[N D_u^2 e^{-(1+r^2)C_u^2}\right]. \tag{2.76}$$

By taking $r = 0$ in (2.76), one gets

$$\frac{\partial}{\partial u}\mathrm{Tr}_s\left[D_u e^{D_u^2}\right] = d\mathrm{Tr}_s\left[\frac{N}{2u}\left(1 + 2D_u^2\right) e^{D_u^2}\right], \tag{2.77}$$

which is exactly [12, (2.32)] (compare also with [12, (3.103)–(3.105)]).

Remark 2.21 Formula (2.77) plays an essential role in [12] in the construction of analytic torsion form. While it can be proved directly as in [12], here we obtain it through purely considerations of η-forms. This suggests that there should be a deep relationship between η and torsion invariants as well as forms.

We now come back to (2.35). We take the derivative with respect to r in it, and then take $r = 0$. What we get is

$$\frac{1}{4\sqrt{u}}\mathrm{Tr}_s\left[\left(v - v\right) e^{D_u^2}\right] = \frac{1}{2u} d\mathrm{Tr}_s\left[N e^{D_u^2}\right]. \tag{2.78}$$

Together with (2.9), we get

$$\mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u} e^{D_u^2}\right] = \frac{1}{2u} d\mathrm{Tr}_s\left[N e^{D_u^2}\right]. \tag{2.79}$$

It is interesting to compare (2.77) and (2.79). In particular, we can rewrite (2.79) as (compare to (3.133))

$$-\frac{\partial}{\partial u}\int_0^1 \mathrm{Tr}_s\left[D_u e^{s^2 D_u^2}\right] ds = \frac{1}{2u} d\mathrm{Tr}_s\left[N e^{D_u^2}\right]. \tag{2.80}$$

By (2.62), (2.63), (2.68) and (2.80), one can give a direct proof of (2.70). As was pointed out in Remark 2.20, this would also lead to a proof of [12, Theorem 2.22]. A comparison like this in the fibration case would be more interesting.

3 Sub-signature operators and a Riemann–Roch formula

In this section, we deal with the fibration case. We will give a new proof of the imaginary part of Theorem 1.1, which is a Riemann–Roch–Grothendieck formula due to Bismut–Lott [12], by computing the adiabatic limits of η invariants of deformed sub-signature operators. We will also prove the real part of Theorem 1.1, in its full generality, by using the same method. Moreover, we will give a natural derivation of the Bismut–Lott analytic torsion form [12] through the transgression of η forms appearing in the adiabatic limit computations.

This Section is organized as follows. In Sect. 3.1, we recall the construction of the Bismut–Lott superconnection introduced in [12]. In Sect. 3.2, we define the sub-signature operator as in [30], as well as a deformation of this operator. In Sect. 3.3, we state the Lichnerowicz type formula for the deformed sub-signature operator. In Sect. 3.4, we state the main technical result of this section, Theorem 3.9, on the adiabatic limit of the η invariants for the deformed sub-signature operators, which will be proved in Sects. 3.5 and 3.6. In Sect. 3.7, we prove Bismut–Lott's formula (1.2) through η invariants. In Sect. 3.8, we prove (1.4) by using Theorem 3.9. In Sect. 3.9, we discuss in detail the relation of our results with Lott's $\mathbb{R}/\mathbb{Z}$ index theory [25]. In Sect. 3.10, we will construct the Bismut–Lott analytic torsion form through the transgression of η forms, which is one of the main points of view of this paper.

3.1 The Bismut–Lott Superconnection

Let $\pi : M \to B$ be a smooth fiber bundle with compact fiber Z of dimension n. We denote by $m = \dim M$, $p = \dim B$. Let TZ be the vertical tangent bundle of the fiber bundle, and let T^*Z be its dual bundle. Let F be a flat complex vector bundle on M and let ∇^F denote its flat connection.

Let $TM = T^H M \oplus TZ$ be a splitting of TM. Let P^{TZ}, $P^{T^H M}$ denote the projection from TM to TZ, $T^H M$. If $U \in TB$, let U^H be the lift of U in $T^H M$, so that $\pi_* U^H = U$.

Let $E = \oplus_{i=0}^{n} E^i$ be the smooth infinite-dimensional $\mathbb{Z}$-graded vector bundle over B whose fiber over $b \in B$ is $C^\infty(Z_b, (\Lambda(T^*Z) \otimes F)|_{Z_b})$. That is

$$C^\infty\left(B, E^i\right) = C^\infty\left(M, \Lambda^i\left(T^*Z\right) \otimes F\right). \tag{3.1}$$

Definition 3.1 For $s \in C^\infty(B, E)$ and U a vector field on B, then the Lie differential L_{U^H} acts on $C^\infty(B, E)$. Let ∇^E be the $\mathbb{Z}$-grading preserving connection on E defined by

$$\nabla^E_U s = L_{U^H} s. \tag{3.2}$$

If U_1, U_2 are vector fields on B, put

$$T(U_1, U_2) = -P^{TZ}\left[U_1^H, U_2^H\right] \in C^\infty(M, TZ). \tag{3.3}$$

We denote by $i_T \in \Lambda^2(B, \mathrm{Hom}(E^\bullet, E^{\bullet-1}))$ the 2-form on B which, to vector fields U_1, U_2 on B, assigns the operation of interior multiplication by $T(U_1, U_2)$ on E.

Let d^Z be the exterior differentiation along fibers. We consider d^Z to be an element of $C^\infty(B, \mathrm{Hom}(E^\bullet, E^{\bullet+1}))$.

The exterior differentiation operator d^M, acting on $\Omega(M,F)=C^\infty(M,\Lambda(T^*M)\otimes F)$, has degree 1 and satisfies $(d^M)^2=0$.

By [12, Proposition 3.4], we have

$$d^M = d^Z + \nabla^E + i_T. \tag{3.4}$$

So d^M is a flat superconnection of total degree 1 on E.

We have

$$\left(d^Z\right)^2 = 0, \quad \left[\nabla^E, d^Z\right] = 0. \tag{3.5}$$

Let h^F be a Hermitian metric on F. Let ∇^{F*} be the adjoint of ∇^F with respect to h^F. Let $\omega(F,h^F)$ and $\nabla^{F,e}$ be the 1-form on M and the connection on F defined as in (2.2), (2.7) respectively.

Let g^{TZ} be a metric on TZ. Let $o(TZ)$ be the orientation bundle of TZ, a flat real line bundle on M. Let dv_Z be the Riemannian volume form on fibers Z associated to the metric g^{TZ} (Here dv_Z is viewed as a section of $\Lambda^{\dim Z}(T^*Z)\otimes o(TZ)$).

Let $\langle\,,\,\rangle_{\Lambda(T^*Z)\otimes F}$ be the metric on $\Lambda(T^*Z)\otimes F$ induced by g^{TZ}, h^F. Then E acquires a Hermitian metric h^E such that for $\alpha,\alpha'\in C^\infty(B,E)$ and $b\in B$,

$$\langle\alpha,\alpha'\rangle_{h^E}(b) = \int_{Z_b}\langle\alpha,\alpha'\rangle_{\Lambda(T^*Z)\otimes F}\, dv_{Z_b}. \tag{3.6}$$

Let $\nabla^{E*}, d^{Z*}, (d^M)^*, (i_T)^*$ be the formal adjoints of ∇^E, d^Z, d^M, i_T with respect to the scalar product $\langle\,,\,\rangle_{h^E}$.

Set

$$D^Z = d^Z + d^{Z*}, \quad \nabla^{E,e} = \frac{1}{2}\left(\nabla^E + \nabla^{E*}\right),$$
$$\omega\left(E,h^E\right) = \nabla^{E*} - \nabla^E. \tag{3.7}$$

Let N_Z be the number operator of E, i.e. N_Z acts by multiplication by k on $C^\infty(M,\Lambda^k(T^*Z)\otimes F)$.

For $u>0$, set

$$C'_u = u^{N_Z/2} d^M u^{-N_Z/2}, \quad C''_u = u^{-N_Z/2}(d^M)^* u^{N_Z/2},$$
$$C_u = \frac{1}{2}\left(C'_u + C''_u\right), \quad D_u = \frac{1}{2}\left(C''_u - C'_u\right). \tag{3.8}$$

Then C''_u is the adjoint of C'_u with respect to h^E. Moreover, C_u is a superconnection on E and D_u is an odd element of $\Omega(B,\mathrm{End}(E))$, and

$$C_u^2 = -D_u^2, \quad [C_u, D_u] = 0. \tag{3.9}$$

Let g^{TB} be a Riemannian metric on TB. Then $g^{TM} = g^{TZ}$ g^{TB} is a metric on TM. Let ^{TM}, ^{TB} denote the corresponding Levi-Civita connections on TM, TB. Put $^{TZ} = P^{TZ}$ ^{TM}, a connection on TZ. As shown in [7, Theorem 1.9], ^{TZ} is independent of the choice of g^{TB}. Then $^{0} =$ ^{TZ} ^{TB} is also a connection on TM.

Let $S =$ ^{TM} Š ^{0}. By [7, Theorem 1.9], $S(\cdot)\cdot, \cdot$ $_{g^{TM}}$ is a tensor independent of g^{TB}. Moreover, for U_1, U_2 TB, X, Y TZ,

$$S\ U_1^H\ X, U_2^H\ _{g^{TM}} = Š\ S\ U_1^H\ U_2^H, X\ _{g^{TM}}$$

$$= S(X)U_1^H, U_2^H\ _{g^{TM}} = \frac{1}{2}\ T\ U_1^H, U_2^H\ , X\ _{g^{TM}}, \quad (3.10)$$

$$S(X)Y, U_1^H\ _{g^{TM}} = Š\ S(X)U_1^H, Y\ _{g^{TM}} = Š\ \frac{1}{2}\ L_{U_1^H} g^{TZ}\ (X, Y),$$

and all other terms are zero.

Let $\{f\ \}^p_{=1}$ be an orthonormal basis of TB, set $\{f\ \}^p_{=1}$ the dual basis of T B. In the following, it's convenient to identify f with f^H.

Let $\{e_i\}^n_{i=1}$ be an orthonormal basis of (TZ, g^{TZ}).

We define a horizontal 1-form k on M by

$$k(f\ ) = Š \sum_i S(e_i)\, e_i, f\ . \quad (3.11)$$

Set

$$c(T) = \frac{1}{2} \sum_{,}\ f\ \ f\ c\ T\ f\ , f\ ,$$

$$c(T) = \frac{1}{2} \sum_{,}\ f\ \ f\ c\ T\ f\ , f\ . \quad (3.12)$$

Let $^{(\ T\ Z)}$ be the connection on $(\ T\ Z)$ induced by ^{TZ}. Let $^{TZ\ F,e}$ be the connection on $(\ T\ Z)$ F induced by $^{(\ T\ Z)}$, ^{F,e}.

By [12, (3.36), (3.37), (3.42)],

$$D^Z = \sum_j c(e_j)\ ^{TZ\ F,e}_{e_j}\ Š\ \frac{1}{2} \sum_j c(e_j)\ \ F, h^F\ (e_j),$$

$$d^Z\ Š\ d^Z = Š \sum_j c(e_j)\ ^{TZ\ F,e}_{e_j} + \frac{1}{2} \sum_j c(e_j)\ \ F, h^F\ (e_j),$$

$$^{E,e} = \ f\ \ ^{TZ\ F,e}_{f} + \frac{1}{2} k(f\ )\ , \quad (3.13)$$

$$E, h^E\ = \ f\ \sum_{i,j} S(e_i)e_j, f\ \ c(e_i)c(e_j) + \ F, h^F\ (f\ )\ .$$

By [12, Proposition 3.9], we get

$$\begin{aligned} C_u &= \frac{\sqrt{u}}{2} D^Z + \nabla^{E,e} - \frac{1}{2\sqrt{u}} c(T), \\ D_u &= \frac{\sqrt{u}}{2} \left(d^{Z*} - d^Z \right) + \frac{1}{2} \left(\omega\left(E, h^E\right) - \frac{1}{2\sqrt{u}} c(T) \right). \end{aligned} \tag{3.14}$$

Let $H^{\bullet}(Z, F|_Z) = \bigoplus_{i=0}^{n} H^i(Z, F|_Z)$ be the $\mathbf{Z}$-graded vector bundle over B whose fiber over $b \in B$ is the cohomology $H(Z_b, F|_{Z_b})$ of the sheaf of locally flat sections of F on Z_b.

By [12, Sect. 3(f)], the flat superconnection d^M induces a canonical flat connection $\nabla^{H(Z,F|_Z)}$ on $H^{\bullet}(Z, F|_Z)$ which preserves the $\mathbf{Z}$-grading and which does not depend on the choice of $T^H M$. In fact, let $\pi : \mathrm{Ker}(d^Z) \to H^{\bullet}(Z, F|_Z)$ be the quotient map. Let s be a smooth section of $H^{\bullet}(Z, F|_Z)$, then locally, there is a smooth section s' of $\mathrm{Ker}(d^Z)$ such that $\pi(s') = s$. By (3.5), $\nabla^{H(Z,F|_Z)} s := \pi(\nabla^E s')$.

By the Hodge theory, there is an isomorphism $H^{\bullet}(Z_b, F|_{Z_b}) \simeq \mathrm{Ker}(D^{Z_b})$ for any $b \in B$. Thus $\mathrm{Ker}(D^{Z_b})$ has locally constant dimension on B. They together form a vector bundle $\mathrm{Ker}(D^Z)$. Now the fiberwise isomorphism induces an isomorphism of the smooth $\mathbf{Z}$-graded vector bundles on B,

$$H^{\bullet}(Z, F|_Z) \simeq \mathrm{Ker}(D^Z). \tag{3.15}$$

Clearly, as a subbubdle of E, $\mathrm{Ker}(D^Z)$ inherits a metric from the scalar product $\langle\ ,\ \rangle_{h^E}$ in (3.6). Let $h^{H(Z,F|_Z)}$ be the corresponding metric on $H^{\bullet}(Z, F|_Z)$ induced by (3.15).

Let P be the orthogonal projection operator from E on $\mathrm{Ker}(D^Z)$ with respect to the Hermitian product (3.6).

Let $(\nabla^{H(Z,F|_Z)})^*$ be the adjoint of $\nabla^{H(Z,F|_Z)}$ with respect to the Hermitian metric $h^{H(Z,F|_Z)}$.

The following result is established in [12, Proposition 3.14].

Proposition 3.2 *The following identities hold*

$$\begin{aligned} \nabla^{H(Z,F|_Z)} &= P \nabla^E P, \qquad \nabla^{H(Z,F|_Z)*} = P \nabla^{E*} P, \\ \omega\left(H(Z, F|_Z), h^{H(Z,F|_Z)}\right) &= P\, \omega\left(E, h^E\right) P. \end{aligned} \tag{3.16}$$

3.2 The sub-signature operator on a fibered manifold

We assume that TB is oriented.

Let (μ, h^μ) be a Hermitian complex vector bundle over B carrying a Hermitian connection ∇^μ.

Let N_B, N_M be the number operators on $\Lambda(T^*B)$, $\Lambda(T^*M)$ respectively, i.e. they act as multiplication by k on $\Lambda^k(T^*B)$, $\Lambda^k(T^*M)$ respectively. Then $N_M = N_B + N_Z$.

Let $\nabla^{\Lambda(T^*M)}$ be the connection on $\Lambda(T^*M)$ canonically induced from ∇^{TM}. Let $\nabla^{\Lambda(T^*M)\otimes\mu\otimes F}$ (resp. $\nabla^{\Lambda(T^*M)\otimes\mu\otimes F,e}$) be the tensor product connection on $\Lambda(T^*M)\otimes\mu\otimes F$ induced by $\nabla^{\Lambda(T^*M)}$, ∇^{μ} and ∇^{F} (resp. $\nabla^{F,e}$).

Let $\{e_a\}_{a=1}^m$ be an orthonormal basis of TM, and its dual basis $\{e^a\}_{a=1}^m$.

Let $\{f_\alpha\}_{\alpha=1}^p$ be an oriented orthonormal basis of TB.

Set

$$
\begin{aligned}
\varepsilon(TB) &= \left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}} c\left(f_1^H\right)\cdots c\left(f_p^H\right),\\
\hat{\varepsilon}(TB) &= \left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}} \hat{c}\left(f_1^H\right)\cdots \hat{c}\left(f_p^H\right),\\
\tau &= (-1)^{N_Z}\,\varepsilon(TB)\,\hat{\varepsilon}(TB).
\end{aligned}
\tag{3.17}
$$

Then the operators $\varepsilon(TB)$, $\hat{\varepsilon}(TB)$, τ act naturally on $\Lambda(T^*M)$. They are self-adjoint and

$$
\begin{aligned}
&\varepsilon(TB)^2 = (-1)^p, \quad \hat{\varepsilon}(TB)^2 = \tau^2 = 1,\\
&\tau = (-1)^p(-1)^{N_M}\,\hat{\varepsilon}(TB)\,\varepsilon(TB) = \varepsilon(TB)\hat{\varepsilon}(TB)(-1)^{N_M}.
\end{aligned}
\tag{3.18}
$$

Let $d^{\mu\otimes F} : \Omega^a(M,\mu\otimes F) \to \Omega^{a+1}(M,\mu\otimes F)$ be the unique extension of ∇^μ, ∇^F which satisfies the Leibniz rule. Let $d^{\mu\otimes F *}$ be the adjoint of $d^{\mu\otimes F}$ with respect to the scalar product $\langle\,,\,\rangle_{\Omega(M,\mu\otimes F)}$ on $\Omega(M,\mu\otimes F)$ induced by g^{TM}, h^μ, h^F as in (3.6).

As in [13, (4.26), (4.27)], we have

$$
\begin{aligned}
d^{\mu\otimes F} &= \sum_a e^a \wedge \nabla_{e_a}^{\Lambda(T^*M)\otimes\mu\otimes F},\\
d^{\mu\otimes F*} &= -\sum_a i_{e_a}\left(\nabla_{e_a}^{\Lambda(T^*M)\otimes\mu\otimes F} + \omega(F,h^F)(e_a)\right).
\end{aligned}
\tag{3.19}
$$

For $r\in\mathbb{R}$, we introduce the following operators as in [30, (1.12)] and (2.19),

$$
\begin{aligned}
D_{sig}^{\mu\otimes F} &= \frac{1}{2}\left(d^{\mu\otimes F} + d^{\mu\otimes F*} + (-1)^{p+1}\tau\left(d^{\mu\otimes F} + d^{\mu\otimes F*}\right)\tau\right),\\
\tilde{D}_{sig}^{\mu\otimes F} &= \frac{1}{2}\left(d^{\mu\otimes F} - d^{\mu\otimes F*} + (-1)^{p+1}\tau\left(d^{\mu\otimes F} - d^{\mu\otimes F*}\right)\tau\right),\\
D_{sig}^{\mu\otimes F}(r) &= D_{sig}^{\mu\otimes F} + \sqrt{-1}\,r\,\tilde{D}_{sig}^{\mu\otimes F}.
\end{aligned}
\tag{3.20}
$$

Let $(D_{sig}^{\mu\otimes F})^*$, $(\tilde{D}_{sig}^{\mu\otimes F})^*$ be the formal adjoints of $D_{sig}^{\mu\otimes F}$, $\tilde{D}_{sig}^{\mu\otimes F}$ with respect to $\langle\,,\,\rangle_{\Omega(M,\mu\otimes F)}$. Then

$$
\begin{aligned}
&\tau D_{sig}^{\mu\otimes F}\tau = (-1)^{p+1}D_{sig}^{\mu\otimes F}, \quad \tau\tilde{D}_{sig}^{\mu\otimes F}\tau = (-1)^{p+1}\tilde{D}_{sig}^{\mu\otimes F},\\
&\left(D_{sig}^{\mu\otimes F}\right)^* = (-1)^{p+1}D_{sig}^{\mu\otimes F}, \quad \left(\tilde{D}_{sig}^{\mu\otimes F}\right)^* = (-1)^{p}\tilde{D}_{sig}^{\mu\otimes F}.
\end{aligned}
\tag{3.21}
$$

Remark 3.3 If $\mu = \mathbb{C}$, then D^F_{sig} is different from the sub-signature operator in [30, (1.12)] (cf. also [31]) by a factor $(\sqrt{-1})^{p(p+1)/2}(-1)^{N_M}$.

Assume now $M = B$, $\mu = F = \mathbb{C}$, then if $p = \dim B$ is odd, $D^{\mathbb{C}}_{sig}$ is exactly the odd Signature operator in [3, (2.1)], [8, (1.38)], and $D^{\mathbb{C}}_{sig} = 0$; if p is even, then $D^{\mathbb{C}}_{sig} = (d + d^*)$ and $D^{\mathbb{C}}_{sig} = 0$.

Following [30], we will rewrite $D_{sig}^{\mu \otimes F}$, $D_{sig}^{\mu \otimes F}$ by using the natural connections. Let $\nabla^{\Lambda(T^* M)}$ be the Hermitian connection on $\Lambda(T^* M)$ defined by (cf. [30, (1.21)])

$$\nabla_X^{\Lambda(T^* M)} = \nabla_X^{\Lambda(T^* M)} - \frac{1}{2}\sum_{\alpha=1}^{p} c\left(P^{TZ} S(X) f_\alpha\right) c(f_\alpha), \quad X \in TM. \tag{3.22}$$

Let ∇^e be the tensor product connection on $\Lambda(T^* M) \otimes \mu \otimes F$ induced by $\nabla^{\Lambda(T^* M)}$, ∇^μ and $\nabla^{F,e}$.

For $r \in \mathbb{R}$, set

$$
\begin{aligned}
D^{\mu \otimes F} &= \sum_{a=1}^{m} c(e_a)\nabla^e_{e_a} - \frac{1}{2}\sum_{i=1}^{n} c(e_i)\,\omega(F, h^F)(e_i),\\
D^{\mu \otimes F} &= -\sum_{i=1}^{n} c(e_i)\nabla^e_{e_i} + \frac{1}{2}\sum_{a=1}^{m} c(e_a)\,\omega(F, h^F)(e_a)\\
&\quad - \frac{1}{4}\sum_{\alpha,\beta=1}^{p} c\left(T(f_\alpha, f_\beta)\right) c(f_\alpha) c(f_\beta),\\
D^{\mu \otimes F}(r) &= D^{\mu \otimes F} + \sqrt{-1}\, r\, D^{\mu \otimes F}.
\end{aligned}
\tag{3.23}
$$

The following result extends [30, Proposition 1.14].

Proposition 3.4

$$D_{sig}^{\mu \otimes F} = D^{\mu \otimes F}, \quad D_{sig}^{\mu \otimes F} = D^{\mu \otimes F}. \tag{3.24}$$

Proof By (3.19),

$$
\begin{aligned}
d^{\mu} + d^{\mu *} &= \sum_{a=1}^{m} c(e_a)\left(\nabla_{e_a}^{\Lambda(T^* M)} \otimes \nabla^{\mu \otimes F,e} - \frac{1}{2}c(e_a)\,\omega(F, h^F)(e_a)\right),\\
d^{\mu} - d^{\mu *} &= \sum_{a=1}^{m} -c(e_a)\left(\nabla_{e_a}^{\Lambda(T^* M)} \otimes \nabla^{\mu \otimes F,e} + \frac{1}{2}c(e_a)\,\omega(F, h^F)(e_a)\right).
\end{aligned}
\tag{3.25}
$$

Recall also that the following equation was gotten in [30, (1.24)] by direct computations,

$$\nabla_X^{\Lambda(T^*M)}(TB) = \sum_{\beta=1}^{p} (-1)^{\frac{p(p+1)}{2}} c(f_1^H)\cdots c(\nabla_X^{TM} f_\beta)\cdots c(f_p^H)$$
$$= -(TB)\sum_{\beta=1}^{p} c(P^{TZ}S(X)f_\beta)c(f_\beta). \qquad (3.26)$$

By (3.10) and (3.17),

$$\frac{1}{2}(-1)^{p+1}\sum_a c(e_a)c(P^{TZ}S(e_a)f)$$
$$= \frac{}{2}\sum_i c(e_i)c(P^{TZ}S(e_i)f) - c(f)c(P^{TZ}S(f)f)$$
$$= \frac{}{2}\sum_i c(e_i)c(P^{TZ}S(e_i)f) - \frac{1}{2}c(T(f,f))c(f). \qquad (3.27)$$

Now (3.24) is a direct consequence of (3.25)–(3.27).

From (3.23), the operators D^F, $D^F(r)$ are formally self-adjoint first order elliptic differential operators, and D^F is a skew-adjoint first order differential operator.

The operator D^F is locally of Dirac type.

By (3.18), (3.21) and (3.24),

$$D^F = (-1)^{p+1}D^F, \qquad D^F = (-1)^{p+1}D^F. \qquad (3.28)$$

3.3 A Lichnerowicz type formula for $D_{\mathrm{sig}}^F(r)$

If $B \in \mathrm{End}(TM)$ is antisymmetric, then the action of B on $\Lambda(T^*M)$ as a derivation (cf. [5, (1.26)]) is given by

$$\sum_{a,b}\langle e_b, Be_a\rangle e^b \wedge i_{e_a} = \frac{1}{4}\sum_{a,b}\langle e_b, Be_a\rangle\left(c(e_a)c(e_b) - \hat{c}(e_a)\hat{c}(e_b)\right). \qquad (3.29)$$

Let $\nabla^{T^HM} = P^{T^HM}\nabla^{TM}$ be the connection on T^HM induced by ∇^{TM}. Let R^{TM}, R^{T^HM}, R^{TZ} be the curvatures of ∇^{TM}, ∇^{T^HM}, ∇^{TZ} respectively. Let K be the scalar curvature of (M, g^{TM}). Let $\nabla^{TM} = \nabla^{T^HM} \oplus \nabla^{TZ}$ be the connection on TM with curvature R^{TM}. Then

$$\nabla^{TM} = \nabla^{TM} + S(\cdot) - P^{T^HM}S(\cdot)P^{T^HM}. \qquad (3.30)$$

Set

$$R^{e} = -\frac{1}{4}\sum_{\alpha,\beta=1}^{p} R^{T^HM} f_\alpha, f_\beta \quad c(f_\alpha)c(f_\beta)$$
$$-\frac{1}{4}\sum_{i,j=1}^{n} R^{TZ} e_i, e_j \quad c(e_i)c(e_j) - \frac{1}{4} \quad F, h^{F} \quad ^{2}. \tag{3.31}$$

We explain now how to get $^{(T M)}$ as a Clifford connection.

Assume temporary that $T M$ is spin.

Let $S(T M)$ be the spinor bundle of $T M$.

By (2.13), $c(\cdot)$ is the Clifford action on $(T M)$ and we have the following isomorphism of Clifford modules,

$$(T M) \quad _{\mathbb{R}} \mathbb{C} \quad S(T M) \quad S(T M) . \tag{3.32}$$

Again by (2.13), $c(\cdot)$ End$(S(T M))$.

Let $^{S(T M)}$ (resp. $^{S(T M)}$) be the Clifford connection on $S(T M)$ induced by $^{T M}$ (resp. $S(T M)$ induced by $^{T M}$) on $T M$ with curvatures $R^{S(T M)}$ (resp. $R^{S(T M)}$).

By (3.22), (3.30) and (3.32), $^{(T M)}$ is the connection on $(T M)$ $_{\mathbb{R}} \mathbb{C}$ induced by $^{S(T M)}$ and $^{S(T M)}$, and

$$R^{S(T M)} = \frac{1}{4}\sum_{a,b=1}^{m} R^{T M} e_a, e_b \quad c(e_a)c(e_b),$$
$$R^{S(T M)} = -\frac{1}{4}\sum_{a,b=1}^{m} R^{T M} e_a, e_b \quad c(e_a)c(e_b). \tag{3.33}$$

But locally, $T M$ is spin, thus by (3.22), (3.31) and (3.33), the curvature of ^{e} is given by

$$^{e\,2} = \frac{1}{4}\sum_{a,b=1}^{m} R^{T M} e_a, e_b \quad c(e_a)c(e_b) + R^{e} + \quad R^{\mu}. \tag{3.34}$$

Let $^{T M \quad F,e}$ (F, h^{F}) be the covariant derivative of (F, h^{F}). Explicitly

$$^{T M \quad F,e}_{e_a} \quad F, h^{F} \quad (e_b)$$
$$= \quad ^{(T M) \quad F}_{e_a} \quad F, h^{F} \quad (e_b) + \frac{1}{2} \quad F, h^{F} \quad ^{2}(e_a, e_b). \tag{3.35}$$

Let Δ^e be the Bochner Laplacian

$$\Delta^e = -\sum_{a=1}^m \left(\left(\nabla^e_{e_a}\right)^2 - \nabla^e_{\nabla^{TM}_{e_a} e_a} \right). \tag{3.36}$$

The following result was proved in [30, Theorem 1.1] based on a direct computation (i.e. applying the Lichnerowicz formula).

Proposition 3.5

$$\begin{aligned} D^{\mu\, F,2} = & -\Delta^e + \frac{K}{4} + \frac{1}{2}\sum_{a,b=1}^m c(e_a)c(e_b)\left(R^e + R^\mu \right)(e_a,e_b) \\ & + \frac{1}{4}\sum_{i=1}^n \left(\omega(F,h^F)(e_i)\right)^2 + \frac{1}{8}\sum_{i,j=1}^n c(e_i)c(e_j)\left(\omega(F,h^F)\right)^2(e_i,e_j) \\ & - \frac{1}{2}\sum_{a=1}^m c(e_a)\sum_{i=1}^n c(e_i)\Big(\left(\nabla^{TM\otimes F,e}_{e_a}\omega(F,h^F)\right)(e_i) \\ & + \omega(F,h^F)\left(S(e_a)e_i\right)\Big). \end{aligned} \tag{3.37}$$

Similarly, for $D^{\mu\, F,2}$, $[D^{\mu\, F}, D^{\mu\, F}]$, we have

Proposition 3.6

$$\begin{aligned} D^{\mu\, F,2} = & -\sum_{i=1}^n \left(\left(\nabla^e_{e_i}\right)^2 - \nabla^e_{\nabla^{TZ}_{e_i} e_i} \right) + \frac{1}{2}\sum_{i,j=1}^n c(e_i)c(e_j)\left(\nabla^e\right)^2(e_i,e_j) \\ & + \frac{1}{4}\sum_{i=1}^n c(e_i)\left[\nabla^e_{e_i}, \sum_{\alpha,\beta=1}^p c\left(T(f_\alpha,f_\beta)\right)c(f_\alpha)c(f_\beta)\right] \\ & + \frac{1}{2}\sum_{\alpha,\beta=1}^p c(f_\alpha)c(f_\beta)\nabla^e_{T(f_\alpha,f_\beta)} \\ & - \frac{1}{2}\sum_{i=1}^n\sum_{a=1}^m c(e_i)c(e_a)\left(\nabla^{TM\otimes F,e}_{e_i}\omega(F,h^F)\right)(e_a) \\ & - \frac{1}{4}\sum_{a=1}^m \left(\omega(F,h^F)(e_a)\right)^2 \\ & + \frac{1}{8}\sum_{a,b=1}^m c(e_a)c(e_b)\left(\omega(F,h^F)\right)^2(e_a,e_b) \\ & + \frac{1}{16}\left(\sum_{\alpha,\beta=1}^p c\left(T(f_\alpha,f_\beta)\right)c(f_\alpha)c(f_\beta)\right)^2, \end{aligned} \tag{3.38}$$

$$
\begin{aligned}
\left[D^{\mu\, F}, D^{\mu\, F}\right] = & -\sum_{a=1}^{m}\sum_{i=1}^{n} c(e_a)c(e_i)\; R^{e} + R^{\mu} + \frac{1}{4}\; F, h^F\;{}^2\; (e_a, e_i) \\
& - \sum_{a=1}^{m}\sum_{i=1}^{n} c(e_a)c(e_i)\; -\frac{1}{2}\; R^{TM}(e_b, e_i)e_b, e_a + {}^{e}_{S(e_a)e_i} \\
& - \sum_{=1}^{p} \; F, h^F\; (f\;)\; {}^{e}_{f} + \frac{1}{2}\; {}^{TM\;\; F,e}_{f}\; F, h^F\; (f\;) \\
& + \frac{1}{2}\sum_{i=1}^{n} \; F, h^F\; (S(e_i)e_i) \\
& + \frac{1}{4}\sum_{,\;=1}^{p} \; F, h^F\; T\; f\;, f\; c(f\;)c\; f \\
& - \frac{1}{4}\sum_{a=1}^{m} c(e_a)\; {}^{e}_{e_a}, \sum_{,\;=1}^{p} c\; T\; f\;, f\; c(f\;)c\; f\; .
\end{aligned}
$$

Proof To simplify the notation, when a subscript index appears two times in a formula, we sum up with this index.

By the discussion at the beginning of Sect. 3.3 and (3.30), we know that for $X \; TM$,

$$
{}^{e}_{X}, c(e_a) = c\; {}^{TM}_{X} e_a \;, \qquad {}^{e}_{X}, c(e_a) = c\; {}^{TM}_{X} e_a \;. \tag{3.39}
$$

From (3.39),

$$
c(e_i)\; {}^{e}_{e_i} c(e_j)\; {}^{e}_{e_j} = \frac{1}{2} c(e_i)c(e_j)\; {}^{e\,2}(e_i, e_j) + {}^{e\;2}_{e_i} - {}^{e}_{{}^{TZ}_{e_i} e_i} . \tag{3.40}
$$

From (3.23) and (3.40), we get the rst equation of (3.38).

We compute now $[D^{\mu\, F}, D^{\mu\, F}]$.

Note that

$$
S(e_a)e_i = {}^{TM}_{e_a} e_i - {}^{TZ}_{e_a} e_i ,\quad c\; {}^{TM}_{e_i} e_a \; {}^{e}_{e_a} = - c(e_a)\; {}^{e}_{{}^{TM}_{e_i} e_a} .
$$

Combining the above equation with (3.39), we get

$$
\begin{aligned}
c(e_a)\; {}^{e}_{e_a}, c(e_i)\; {}^{e}_{e_i} = & \; c(e_a)c(e_i)\; {}^{e}_{e_a}\, {}^{e}_{e_i} - {}^{e}_{e_i}\, {}^{e}_{e_a} \\
& + c(e_a)c\; {}^{TZ}_{e_a} e_i \; {}^{e}_{e_i} + c(e_i)c\; {}^{TM}_{e_i} e_a \; {}^{e}_{e_a} \\
= & \; c(e_a)c(e_i)\; {}^{e\,2}(e_a, e_i) + c(e_a)c(e_i)\; {}^{e}_{S(e_a)e_i} .
\end{aligned} \tag{3.41}
$$

Recall that for $X, Y, Z, W \quad TM$, we have

$$R^{TM}(X,Y)Z, W = R^{TM}(Z,W)X, Y ,$$
$$R^{TM}(X,Y)Z + R^{TM}(Y,Z)X + R^{TM}(Z,X)Y = 0. \tag{3.42}$$

From (3.42), we get

$$R^{TM}(e_a, e_i)e_b, e_c \; c(e_a)c(e_b)c(e_c) = -2 \; R^{TM}(e_a, e_i)e_a, e_c \; c(e_c). \tag{3.43}$$

Thus by (3.34), (3.41) and (3.43), we get

$$c(e_a) \; {}^{e}_{e_a}, c(e_i) \; {}^{e}_{e_i} = \frac{1}{2} R^{TM}(e_a, e_i)e_a, e_c \; c(e_i)c(e_c)$$
$$+ c(e_a)c(e_i) \quad R^{e} + \quad R^{\mu} + \frac{1}{4} \quad F, h^{F} \quad ^{2} \quad (e_a, e_i)$$
$$+ c(e_a)c(e_i) \quad {}^{e}_{S(e_a)e_i} . \tag{3.44}$$

Note that by (3.35)

$${}^{TM}_{e_a} \; {}^{F,e} \quad F, h^{F} \quad (e_b) - {}^{TM}_{e_b} \; {}^{F,e} \quad F, h^{F} \quad (e_a)$$
$$= \quad ^{(TM)} \; ^{F,e} \quad F, h^{F} \quad (e_a, e_b) = 0. \tag{3.45}$$

Thus

$$c(e_a) \; {}^{e}_{e_a}, c(e_b) \quad F, h^{F} \quad (e_b)$$
$$= -2 \quad F, h^{F} \quad (e_a) \; {}^{e}_{e_a} - \quad {}^{TM}_{e_a} \; {}^{F,e} \quad F, h^{F} \quad (e_a). \tag{3.46}$$

By

$${}^{F,e}_{e_a} \quad F, h^{F} \quad (e_j) = {}^{TM}_{e_a} \; {}^{F,e} \quad F, h^{F} \quad (e_j) + \quad F, h^{F} \quad {}^{TM}_{e_a} e_j \quad ,$$

(2.13), (3.10), (3.39) and (3.45),

$$c(e_i) \; {}^{e}_{e_i}, c(e_j) \quad F, h^{F} \quad (e_j)$$
$$= 2 \quad F, h^{F} \quad (e_i) \; {}^{e}_{e_i}$$
$$+ c(e_i)c(e_j) \; {}^{F,e}_{e_i} \quad F, h^{F} \quad (e_j) + c(e_i)c \quad {}^{TZ}_{e_i} e_j \quad F, h^{F} \quad (e_j)$$

$$= 2 \quad F, h^F \quad (e_i) \quad {}^{e}_{e_i} + c(e_i)c(e_j) \quad {}^{TM \quad F,e}_{e_i} \quad F, h^F \quad (e_j)$$

$$+ c(e_i)c(e_j) \quad F, h^F \quad (S(e_i)e_j)$$

$$= 2 \quad F, h^F \quad (e_i) \quad {}^{e}_{e_i} + \quad {}^{TM \quad F,e}_{e_i} \quad F, h^F \quad (e_i) + \quad F, h^F \quad (S(e_i)e_i). \tag{3.47}$$

We have also

$$c(e_i), c \quad T \quad f \ , f \quad c(f\ )c \quad f \quad = 2\ T \quad f \ , f \quad , e_i \quad c(f\ )c \quad f \quad . \tag{3.48}$$

From (3.23), (3.34), (3.44)–(3.48), we get the second equation of (3.38).

To conclude this subsection, we state the following formula, which is a consequence of (3.22) and will be used in a later occasion.

$$ {}^{e}_{e_a}, c \quad T \quad f \ , f \quad c(f\ )c \quad f \quad = \quad {}^{(\ TM)}_{e_a}, c \quad T \quad f \ , f \quad c(f\ )c \quad f \quad . \tag{3.49}$$

3.4 The invariant for $D_{sig}^{\mu \quad F}(r)$

In this section, we assume that B is a closed oriented compact manifold and $p + \dim B$ is odd.

By (3.20), $D_{sig}^{\mu \quad F}(r)$ preserves the $\mathbf{Z}_2$-grading on $(\ M, \quad \mu \quad F)$ induced by $(-1)^{N_M}$. We denote by $D_{sig,e}^{\mu \quad F}(r)$ the restriction of $D_{sig}^{\mu \quad F}(r)$ on ${}^{even}(M, \mu \quad F)$.

Let $\bar{}(\ D_{sig,e}^{\mu \quad F}(r))$ denote the associated reduced invariant as in (2.24). We will omit the notion of F when $F = \mathbf{C}$ is the trivial complex line bundle carrying the trivial metric and connection.

Definition 3.7 Let $(\ M/B, \mu, \ F, r) \quad \mathbf{R}/\mathbf{Z}$ be defined by

$$(\ M/B, \mu, \ F, r) = \bar{} \quad D_{sig,e}^{\mu \quad F}(r) \quad - \mathrm{rk}(F) \bar{} \quad D_{sig,e}^{\mu}(r) \quad \mathrm{mod}\, \mathbf{Z}. \tag{3.50}$$

When $(F, \quad {}^F, h^F)$ is unitary and $M = B$, $2\ (\ M/B, \mathbf{C}, F, 0) \quad \mathbf{R}/\mathbf{Z}$ is the mod $\mathbf{Z}$ part of the -invariant $(\ M, F)$ associated to $D_{sig,e}^{\mathbf{C}}$ and F in the sense of [3,4], where

$$(\ M, F) = \quad D_{sig,e}^{F} \ - \mathrm{rk}(F) \quad D_{sig,e}^{\mathbf{C}} \quad . \tag{3.51}$$

Theorem 3.8 (i) If n is odd, then $\bar{}(\ D_{sig,e}^{\mu \quad F}(r)) \quad \mathbf{R}/\mathbf{Z}$ does not depend on $(g^{TB}, g^{TZ}, h^{\mu}, \quad {}^{\mu})$ and h^F.

(ii) The number $(\ M/B, \mu, \ F, r)$ does not depend on $(g^{TB}, g^{TZ}, h^{\mu}, \quad {}^{\mu})$ and h^F.

For any > 0, let $D^{\mu\ \ F}_{sig,e,}(r)$ be the operator obtained above by replacing g^{TB} to $\frac{1}{}g^{TB}$.

The following result is the main technical result of this paper, which generalizes [30, Theorem 0.2].

Theorem 3.9 We have the following identity in R/Z,

$$\lim_{0} \bar{\ } \ D^{\mu\ \ F}_{sig,e,}(r) = \bar{\ }\ D^{\mu\ \ H(Z,F|_Z)}_{sig,e}(r) := \sum_{i=0}^{n} (-1)^i \bar{\ }\ D^{\mu\ \ H^i(Z,F|_Z)}_{sig,e}(r) \ . \tag{3.52}$$

By applying the previous constructions to the special case with $M = B$, one constructs a series of smooth invariants $(B, \mu, H^i(Z; F|_Z), r)$, $0 \quad i \quad n$. They are the (generalized) -invariants associated to twisted Signature operators on B.

Corollary 3.10 (i) If $n = \dim Z$ is odd, then

$$\bar{\ }\ D^{\mu\ \ F}_{sig,e}(r) = \bar{\ }\ D^{\mu\ \ H(Z,F|_Z)}_{sig,e}(r) \quad \text{in } R/Z. \tag{3.53}$$

(ii) In general, the following identity holds in R/Z,

$$(M/B, \mu, F, r) = \sum_{i=0}^{n} (-1)^i \quad B, \mu, H^i(Z, F|_Z), r$$

$$-\operatorname{rk}(F) \quad \sum_{i=0}^{n} (-1)^i \quad B, \mu, H^i(Z, C|_Z), r \ . \tag{3.54}$$

3.5 A proof of Theorem 3.8

We will consider a smooth path of data with parameter s [0, 1] and apply the proof of the Atiyah–Patodi–Singer index theorem [2] on $M \times [0, 1]$. Here we need to compute the local index density on $M \times [0, 1]$, and we conclude our result by analyzing our local formula.

Let $g^{TB}_s, g^{TZ}_s, T^H_s M, h^F_s, h^{\mu}_s, \ ^{\mu}_s$ $(s \quad [0, 1])$ be a smooth family of the objects as in Sect. 3.1. In order to apply the Atiyah–Patodi–Singer index theorem strictly, without loss of generality we assume that $g^{TB}_s, g^{TZ}_s, T^H_s M, h^F_s, h^{\mu}_s, \ ^{\mu}_s$ do not depend on s near the end points 0 and 1.

Consider the fibration $: M = M \times R \quad B = B \times R$.

Let $_1 : M \quad M$, $_B : B \quad B$ be the natural projections.

We define $T^H M|_{M\times\{s\}} = T^H_s M \quad R$, $g^{TB}|_{B\times\{s\}} = g^{TB}_s \quad ds^2$, $h^{\ B\mu}|_{B\times\{s\}} = h^{\mu}_s$, $h^{\ _1 F}|_{B\times\{s\}} = h^F_s$.

Clearly,

$$^{B\mu} = \ ^{\mu}_s + ds \quad \overline{\ \ s} + \frac{1}{2} h^{\mu}_s{}^{-1} \frac{h^{\mu}_s}{s} \tag{3.55}$$

is a Hermitian connection on $(\Lambda_B \mu, h^{B\mu})$ with curvature $R^{B\mu}$.

We orient TB as follow: if $\{f\}^{p}_{=1}$ is an oriented orthonormal basis of TB, then the orientation of TB is defined by $f^1 \wedge \cdots \wedge f^p \wedge ds$.

We denote by $f_{p+1} = \frac{}{s}$ and

$$= (-1)^{N_Z} \quad TB \ . \tag{3.56}$$

Now all the constructions in Sects. 3.2, 3.3 work well for the fibration .

Let $_{\pm}(M, (\quad {}_1)\ \mu \quad {}_1 F)$ be the ± 1 eigenspaces of in $(M, (\quad {}_1)\ \mu \quad {}_1 F)$. Then by (3.21), (3.28), $D_{sig}^{(\quad {}_1)\ \mu \quad {}_1 F}(r)$ changes the Z_2-grading induced by .

For any $u > 0$, let $P_u(x, y)$ be the smooth kernel of $\exp(-u(D^{(\quad {}_1)\ \mu \quad {}_1 F}(r))^2)$ with respect to the Riemannian volume form $dv_M(y)$.

For $x_0 \quad M$, let $dv_{T_{x_0}M}$ be the Riemannian volume form on $(T_{x_0}M, g^{T_{x_0}M})$.

For $U \quad T_{x_0}M$, let $_U$ be the ordinary derivative in direction U.

For $y = (y_1, \ldots, y_{m+1}) \quad \mathbb{R}^{m+1}$, we identify y as $\sum_{a=1}^{m+1} y_a e_a$ as a vector in $T_{x_0}M$. Set

$$L_{x_0}(r) = - \quad 1 + r^2 \sum_{i=1}^{n} \quad {}_{e_i} + \frac{1}{4} R^{TM}_{x_0} y, e_i \quad {}^2$$
$$- \sum_{=1}^{p+1} \quad {}_{f} + \frac{1}{4} R^{TM}_{x_0} y, f \quad {}^2 - \frac{1}{4} \sum_{, =1}^{p+1} R^{THM}_{x_0} f, f \quad c(f)c \quad f$$
$$- \frac{1}{4} \sum_{i,j=1}^{n} R^{TZ}_{x_0} e_i, e_j + r^2 R^{TM}_{x_0} e_i, e_j \quad c(e_i)c(e_j), \tag{3.57}$$
$$L_{3,x_0} = - \sum_{i=1}^{n} \sum_{a=1}^{m+1} e^a c(e_i) \quad {}_{S_{x_0}(e_a)e_i} + \frac{1}{4} R^{TM}_{x_0} y, S_{x_0}(e_a)e_i \quad .$$

Let $\exp(-tL_{x_0}(r))(y, y)$, $\exp(-t(L_{x_0}(r) + \overline{-1} r L_{3,x_0}))(y, y)$, $(y, y \quad \mathbb{R}^{m+1})$ be the smooth kernels of $\exp(-tL_{x_0}(r))$, $\exp(-t(L_{x_0}(r) + \overline{-1} r L_{3,x_0}))$ associated to $dv_{T_{x_0}M}(y)$ respectively.

Proposition 3.11 For $x_0 \quad M$, one has

$$\lim_{u \quad 0} \mathrm{Tr}[\quad P_u(x_0, x_0)] = (-1)^{\frac{m(m+1)}{2} + p + m} 2^{m+1} \mathrm{rk}(F)$$
$$TB \ \exp - \ L_{x_0}(r) + \overline{-1} r L_{3,x_0} \quad (0, 0) \mathrm{Tr}|_{\mu} \ \exp -R^{B\mu} \quad , \tag{3.58}$$

where $^{!}$ means the coefficient of $e^{1} \cdots e^{m+1} c(e_1)\cdots c(e_{m+1})$ in

$$TB \ \exp\left(- L_{x_0}(r) + \overline{-1} r L_{3,x_0}\right)(0,0) \mathrm{Tr}|_{\mu} \exp\left(- R^{B\mu}\right).$$

Proof At first, by [13, Proposition 4.9], among the monomials in terms of $c(e_a)$'s and $c(e_a)$'s, only $c(e_1)c(e_1)\cdots c(e_{m+1})c(e_{m+1})$ has a nonzero supertrace with the Z_2-grading on $(T M)$ defined by $(-1)^{N_M}$. Moreover,

$$\mathrm{Tr}\ (-1)^{N_M} c(e_1)c(e_1)\cdots c(e_{m+1})c(e_{m+1}) = (-2)^{m+1}. \tag{3.59}$$

In view of (3.59), to compute the local index, it is convenient to use the rescalings $e_a \to \frac{1}{\sqrt{u}} e_a$, $c(e_a) \to \frac{1}{\sqrt{u}} e^a - \sqrt{u}\, i_{e_a}$, $c(e_a) \to c(e_a)$ and $y_a \to \sqrt{u}\, y_a$.

We denote by $L_{1,u}$, $L_{2,u}$, $L_{3,u}$ the operators obtained from $uD^{\mu F,2}$, $uD^{\mu F,2}$, $u[D^{\mu F}, D^{\mu F}]$ after the above rescalings.

By Propositions 3.5, 3.6, (3.49) and (3.57), as $u \to 0^{+}$,

$$L_{1,u} \to - \sum_{a=1}^{m+1} \left(e_a + \frac{1}{4} R^{TM}_{x_0} y, e_a \right)^2 + R^{e}_{x_0} + R^{B\mu}_{x_0}.$$

$$L_{2,u} \to \sum_{i=1}^{n} \left(e_i + \frac{1}{4} R^{TM}_{x_0} y, e_i \right)^2 + \frac{1}{4} \sum_{i,j=1}^{n} R^{TM}_{x_0} e_i, e_j \ c(e_i)c(e_j) + \frac{1}{4} \left({}_1 F, h\, {}_1^{F} \right)^2_{x_0}, \tag{3.60}$$

$$L_{3,u} \to L_{3,x_0}.$$

Thus after rescaling, the operator $u(D^{\mu F}(r))^2$ has the limit

$$L_{x_0}(r) + \overline{-1} r L_{3,x_0} + R^{B\mu}_{x_0} - \frac{1+r^2}{4} \left({}_1 F, h\, {}_1^{F} \right)^2_{x_0}. \tag{3.61}$$

By (3.18), (3.59), (3.61) and by proceeding the standard local index technique (cf. [5]), we get

$$\lim_{\to 0} \mathrm{Tr}\,[\ P_u(x_0,x_0)] = (-1)^{\frac{m(m+1)}{2}} (-2)^{m+1} (-1)^{p+1}$$
$$TB \ \exp\left(- L_{x_0}(r) + \overline{-1} r L_{3,x_0}\right)(0,0)\, \mathrm{Tr}|_{\mu} \exp\left(- R^{B\mu}\right)$$
$$\cdot \mathrm{Tr}|_{F} \ \exp\left(\frac{1+r^2}{4} \left({}_1 F, h\, {}_1^{F} \right)^2 \right). \tag{3.62}$$

Now note that (cf. [12], (3.77)])

$$\mathrm{Tr}|_F \quad \exp \quad \frac{1+r^2}{4} \qquad {}_1 F, h \; {}_1 F \quad {}^2 \quad = \mathrm{rk}(F). \tag{3.63}$$

The proof of Proposition 3.11 is completed.

Proof of Theorem 3.8 From (3.24), Proposition 3.11 and the Atiyah–Patodi–Singer index theorem [2, Theorem 3.10], one gets as in [30, (1.54)] the following mod $\mathbb{Z}$ variation formula of invariants,

$$- \quad D^{\mu \quad F}_{sig,e,0}(r) \quad - \quad - \quad D^{\mu \quad F}_{sig,e,1}(r) \quad = (-1)^{\frac{m(m+1)}{2}+p+m} 2^m \mathrm{rk}(F)$$

$$T B \quad \exp \quad - \quad L_{x_0}(r) + \quad \overline{-1} r L_{3,x_0} \quad (0,0) \mathrm{Tr}|_\mu \quad \exp \quad - R^{B^\mu} \quad .$$

$M \times [0,1]$

$$\tag{3.64}$$

Set ${}_k = \{(t_1, \ldots, t_k) | 0 \quad t_1 \quad \cdots \quad t_k \quad 1\} \quad R^k$, and

$$I_k(t) = \quad e^{-(t - t_k) L_{x_0}(r)} \; \overline{-1} r L_{3,x_0} e^{-(t_k - t_{k-1}) L_{x_0}(r)} \cdots \; \overline{-1} r L_{3,x_0} e^{-t_1 L_{x_0}(r)} dt_1 \cdots dt_k.$$

$t \quad k$

$$\tag{3.65}$$

By (3.57), $I_k(t) = 0$ for $k > m+1$.

By the Volterra expansion formula (cf. [5, Sect. 2.4]), we have

$$\exp(-t(L_{x_0}(r) + \overline{-1} r L_{3,x_0})) = \exp(-t L_{x_0}(r)) + \sum_{k=1}^{m+1} (-1)^k I_k(t). \tag{3.66}$$

Now by the uniqueness of the heat kernel, we get

$$\exp(-t L_{x_0}(r))(x, y) = \exp(-t L_{x_0}(r))(-x, -y).$$

By (3.65) and observe that $L_{x_0}(r)$ has even degree on the Clifford variables $c(e_i)$ (resp. $c(f)$), we know that the odd degree part on the Clifford variables $c(e_i)$ and $c(f)$ in $\exp(-t(L_{x_0}(r) + \overline{-1} r L_{3,x_0}))$ is $- \sum_{k=1}^{m} I_{2k+1}(t)$ and $I_{2k+1}(t)(x, y) = - I_{2k+1}(t)(-x, -y)$. Thus we know

$$T B \quad \exp \quad - \quad L_{x_0}(r) + \quad \overline{-1} r L_{3,x_0} \quad (0,0) \mathrm{Tr}|_\mu \quad \exp \quad - R^{B^\mu}$$

is zero if n is odd. Thus we get the rst part of Theorem 3.8.

On the other hand, when dim Z is even, from (3.64) and its application to the trivial complex line bundle case, we get the second part of Theorem 3.8 by subtraction.

Thus the proof of Theorem 3.8 is completed.

3.6 A proof of Theorem 3.9

We will apply the adiabatic limit techniques developed by Bismut–Cheeger [10] and Dai [20], i.e., study the limit of the corresponding objects associated to $\frac{1}{\varepsilon} g^{TB}$ as $\varepsilon \to 0$. The general strategy has already been explained in detail in their works, here, we only need to compute the adiabatic limit of the trace of certain heat kernels when $\varepsilon \to 0$.

We will distinguish the objects in Sect. 3.2 associated to $\frac{1}{\varepsilon} g^{TB}$, instead of g^{TB}, by adding a subscript.

By (2.12) and (3.17), for $u_1, u_2 \in \mathbb{R}$, $c = c$ or c,

$$N_B/2 \; c \quad u_1 \; \bar{} \; f + u_2 e_i \quad ŠN_B/2 = u_1 c(f) + u_2 c(e_i), \qquad N_B/2 \quad ŠN_B/2 = \;. \tag{3.67}$$

Denote by $D_{S,}$ μ F $=$ $N_B/2$ D μ F $ŠN_B/2$, similarly, we define $D_{S,}$ μ F and $D_{S,}$ μ $F(r)$.

Let $:(T B) \quad (T B)$ be defined by $= (2 \quad \overline{Š1})^{Š\deg/2}$.

Since we have twisted a vector bundle μ on B, the superconnection in Sect. 3.1 should be modified accordingly.

Let E μ, e be the connection on $E \quad \mu$ induced by E,e and μ. Denote by C_u^{μ} the superconnection on $\mu \quad E$ defined by replacing E,e in (3.14) by $E \quad \mu,e$. All other operators in (3.14) extend naturally on $E \quad \mu$. Let be the connection on $(T M) \quad \mu \quad F$ induced from $0 = \quad TB \quad TZ, \quad \mu, \quad F,e$.

Theorem 3.12 For any $u > 0$, one has

$$\frac{1}{} \lim_{0} \mathrm{Tr} \; D_{sig,e,}^{\mu \; F}(r) \exp\; Šu D_{sig,e,}^{\mu \; F}(r)^2 \; = \int_B L \; T B, \; TB \; \mathrm{ch} \; \mu, h^{\mu}$$

$$\cdot \; \frac{1}{2 \quad \overline{Š1}}^{1/2} \; \mathrm{Tr}_s \; 2 \; \bar{u} \frac{}{u} \; C_{4u} + \overline{Š1} r D_{4u} \; \exp \; Š \; 1 + r^2 \; C_{4u}^2 \quad , \tag{3.68}$$

where the Tr_s on E is deÞned by the $\mathbb{Z}_2$-grading induced from $(Š1)^{N_Z}$.

Proof Following [11] and [10], let z be an odd Grassmannian variable which anti-commutes with $c(e_a)$'s and $c(e_a)$'s.

As in [10, (4.54)], if A, B are of trace class in $\mathrm{End}(\;(M, \quad \mu \quad F))$, set

$$\mathrm{Tr}^z[A + zB] = \mathrm{Tr}[B]. \tag{3.69}$$

One finds as in [11] and [10, (4.55)] that by (3.17), (3.24), (3.28) and (3.67),

$$\bar{u}\mathrm{Tr} \; D_{sig,e,}^{\mu \; F}(r) \exp(Šu D_{sig,e,}^{\mu \; F}(r)^2)$$

$$= \frac{1}{2} \; \bar{u}\mathrm{Tr} \quad D \;\; \mu \;\; F(r) \exp(Šu D \;\; \mu \;\; F(r)^2)$$

$$= Š\frac{1}{2}\mathrm{Tr}^{z} \quad \exp(Šu D^{\mu \quad F}(r)^2 + z \quad \bar{u} D^{\mu \quad F}(r))$$

$$= Š\frac{1}{2}\mathrm{Tr}^{z} \quad \exp(Šu D_{s,}^{\mu \quad F}(r)^2 + z \quad \bar{u} D_{s,}^{\mu \quad F}(r)) \quad . \tag{3.70}$$

In [30, Proposition 2.2], Zhang formulated a Lichnerowicz type formula for $u(D_{s,}^{\mu \quad F})^2 Š z \quad \bar{u} D_{s,}^{\mu \quad F}$ which is obtained from (3.10) and (3.13).

The corresponding degenerate term as 0 is

$$Šu \qquad {}_{f} + \frac{\bar{\ }}{2} \sum_{i,} S(f)e_i, f \quad c(e_i)c(f) + \frac{zc(f)}{2\bar{\bar{u}}}^{2} \quad . \tag{3.71}$$

Note that by (3.10), $S(\cdot)e_i = S(\cdot)e_i$.

On the other hand, from (3.10), (3.23), (3.67) and Proposition 3.6, it is easy to see that for the operators $(D_{s,}^{\mu \quad F})^2$, $[D_{s,}^{\mu \quad F}, D_{s,}^{\mu \quad F}]$, there is no second order derivative on ${}_{f}$ and all the other terms converge as 0. Thus, the only possible singular term in the local index computation appears in (3.71).

To cancel this singular term in (3.71), one can proceed as in [10, 30]. Here we will give another argument as in [6, Sect. 7], [26, Sect. 7].

We x b_0 B. For > 0 small enough, we can identify the ball $B_0( )$ $T_{b_0}B$ with center 0 and radius to the ball in B by using the exponential map.

Let be the connection on (C(z)) (T B) on B de ned by

$$(C(z)) \quad (T B) = {}_{.}(T B) + \frac{zc(\cdot)}{2\bar{\bar{u}}}. \tag{3.72}$$

Then by (3.29) and (3.72),

$$^{2} = \frac{1}{4} \sum_{,} R^{TB} f \ , f \quad c(f)c(f) Š c(f)c(f) \quad . \tag{3.73}$$

Let be the connection on

$$(C(z)) \quad (T M) \qquad \mu \quad F \qquad (C(z)) \quad (T B) \quad (T Z) \qquad \mu \quad F$$

induced by , ^{TZ}, ${}^{\mu}$ and ^{F,e}.

For y $T_{b_0}B$ suf ciently close to b_0, we lift horizontally the path t R_+ ty into path t R_+ x_t M, with x_t Z_{ty}, $\frac{dx}{dt}$ $T^H M$.

For x_0 Z_0, we identify ((C(z)) (T M) μ $F)_{x_t}$ to

$$((C(z)) \quad (T B))_{b_0} \quad ((T Z) \qquad \mu \quad F)_{x_0}$$

by parallel transport along the curve t x_t Z_{ty} with respect to the connection .

For y Tb$_0$Y, set

$$\mathcal{H} = -\quad {}_{f} + \frac{1}{4} R^{TB}_{b_0} y, f \quad {}^{2} - \frac{1}{4} \quad R^{TB}_{b_0} f , f \quad c(f)c(f). \tag{3.74}$$

Now we do the following Getzler rescaling: $\sqrt{u}\, y$, $_f$ $\frac{1}{\sqrt{u}}$ $_f$ and c(f) $\frac{1}{\sqrt{u}}$ f $-$ $\sqrt{u}$ i $_f$.

Recall that S(·)e$_i$ = $-$S(·)e$_i$. By (3.9), (3.10), (3.23), (3.29), (3.71), (3.73), [30, Proposition 2.2], and by proceeding similarly as in [10, (4.69), (4.70)], the rescaled operator $\mathcal{L}_{u,b_0}(r, \;)$ obtained from $u D^{\mu\ F,2}_{S,}(r) - z \sqrt{u} D^{\mu\ F}_{S,}(r)$ converges as 0, to

$$\mathcal{L}_{u,b_0}(r) = \mathcal{H} + (1 + r^2)(C^{\mu}_{4u})^2$$
$$- z \quad \sqrt{u} D^Z + \frac{c(T)}{4\sqrt{u}} + \quad \sqrt{-1} r \quad \sqrt{u}(d^Z - d^Z) + \frac{c(T)}{4\sqrt{u}} \quad . \tag{3.75}$$

In fact, when r = 0, this follows from [30, (2.41)]; now by (3.10), (3.23) and Proposition 3.6, we find that (3.75) holds in general.

Also from (3.14),

$$C^{\mu\ 2}_{u} = C^2_u + R^{\mu}. \tag{3.76}$$

We denote by $\overset{!}{\ }$ ^{B} $=$ ^{Max} with (T M), is a linear combination of c(f$_{i_1}$) · · ·c(f$_{i_l}$), (i$_1$ < · · · < i$_l$) and ^{Max} is the coefficient of c(f$_1$) · · ·c(f$_p$) in .

By (3.18), (3.59) for (T B), (3.70) and (3.75), we get

$$\lim_{0} \mathrm{Tr} \quad D^{\mu\ F}_{sig,e,}(r) \exp -u D^{\mu\ F}_{sig,e,}(r)^2$$
$$= - \frac{1}{2\sqrt{u}} (-1)^p (-2)^p (-1)^{\frac{p(p-1)}{2}} \ \text{"\#\$} \int_B \ ^B \mathrm{Tr}_s \ \exp -\mathcal{L}_{u,b_0}(r) \ \text{\%\&'}^{z}, \tag{3.77}$$

where $(-1)^p$ is from (3.18), $(-2)^p(-1)^{\frac{p(p-1)}{2}}$ is from (3.59), $\{\ \}^z$ is the coefficient of z.

From (3.76) and (3.77), as in [30, (2.43)], we get

$$\lim_{0} \mathrm{Tr} \quad D^{\mu\ F}_{sig,e,}(r) \exp -u D^{\mu\ F}_{sig,e,}(r)^2$$
$$= - \frac{1}{2}(-1)^{\frac{p(p+1)}{2}+p} \quad \frac{1}{\ } ^{p/2} \int_B \det{}^{1/2} \quad \frac{R^{TB}/2}{\sinh(R^{TB}/2)} \quad \mathrm{Tr} \quad e^{-R^{\mu}}$$

$$\times \mathrm{Tr}_s\left[2u\frac{\partial}{\partial u}\left(C_{4u}+\sqrt{-1}\,r D_{4u}\right)\exp\left(-\left(1+r^2\right)C_{4u}^2\right)\right]$$
$$\times \int_B \widehat{A}(TB)\exp\left(-\frac{1}{4}\left\langle R^{TB} f_\alpha, f_\beta\right\rangle c(f_\alpha)c(f_\beta)\right). \tag{3.78}$$

From (3.17), the last term of (3.78) is $(\sqrt{-1})^{\frac{p(p+1)}{2}}\det^{1/2}(\cosh(R^{TB}/2))$ as in [30, (2.44)]. Thus we get (3.68) as p is odd.

The following Lemma tells us that the right hand side of (3.68) is zero.

Lemma 3.13

$$\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}\,r D_u\right)\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]$$
$$=\frac{\sqrt{-1}\,r}{2u}\frac{\partial}{\partial r}\mathrm{Tr}_s\left[N_Z\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]. \tag{3.79}$$

Proof By (3.9),

$$\left(C_u+\sqrt{-1}\,r D_u\right)^2=\left(1+r^2\right)C_u^2. \tag{3.80}$$

By (3.14), we have (cf. also [26, p. 19])

$$2u\frac{\partial C_u}{\partial u}=-[N_Z, D_u],\quad 2u\frac{\partial D_u}{\partial u}=-[N_Z, C_u]. \tag{3.81}$$

Thus by (3.80) and (3.81)

$$\mathrm{Tr}_s\left[\frac{\partial}{\partial u}\left(C_u+\sqrt{-1}\,r D_u\right)\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]$$
$$=-\frac{\sqrt{-1}}{2u}\mathrm{Tr}_s\left[\left([N_Z, D_u]+\sqrt{-1}\,r[N_Z, C_u]\right)\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]$$
$$=\frac{\sqrt{-1}\,r}{2u}\mathrm{Tr}_s\left[[C_u, N_Z]\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]$$
$$=\frac{\sqrt{-1}\,r}{2u}\frac{\partial}{\partial r}\mathrm{Tr}_s\left[N_Z\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right], \tag{3.82}$$

as

$$\mathrm{Tr}_s\left[[N_Z, D_u]\exp\left(-\left(C_u+\sqrt{-1}\,r D_u\right)^2\right)\right]=\mathrm{Tr}_s\left[\left[N_Z, D_u\exp\left(-\left(1+r^2\right)D_u^2\right)\right]\right]=0.$$

Proof of Theorem 3.9 From Lemma 3.13, we know that the right side of (3.68) is zero. By mimicking the argument in [20], we get Theorem 3.9 (compare also with [25, p. 298] for the precise counting of the mod Z term).

3.7 invariant and Bismut–Lott theorem

We prove the Bismut–Lott formula (1.2) in this subsection.

Let $(\widehat{T Z})$ be another copy of $(T Z)$.

For $(T Z)$, we denote by $(\widehat{T Z})$ the copy of .

The Berezin integral $\int^{B} : (T M) \quad (\widehat{T Z}) \quad (T M) \quad o(T Z)$ is de ned by $\int^{B}$ $i_{e_1} \cdots i_{e_n}$ for $(T M)$.

We de ne also the ber-wise integral $\int_Z$ by: for $C(B, (T B))$, $C(M, (T Z) \quad o(T Z))$,

$$= \quad . \qquad Z \qquad Z \tag{3.83}$$

Set

$$R^{T Z} = \frac{1}{2} \sum_{i,j=1}^{n} e_i, R^{T Z} e_j \; e^i \; e^j \quad {}^2(T M) \quad \widehat{{}^2(T Z)},$$
$$e(T Z, {}^{T Z}) = (Š1)^{\frac{n(n+1)}{2}} \; Š^{\frac{n}{2}} \; {}^{B} \exp Š\frac{1}{2} R^{T Z} \; . \tag{3.84}$$

The form $e(T Z, {}^{T Z})$ is the Chern-Weil representative of the Euler class $e(T Z)$ of $T Z$. Certainly, $e(T Z, {}^{T Z}) = 0$ if $n = \dim Z$ is odd.

By the standard variation formula for the reduced eta invariants (cf. [1, Proposition 2.8]), we nd that for any $r \quad R$,

$$\frac{}{r} \bar{\;} \; D^{\mu \; F}_{sig,e}(r)$$
$$= \lim_{u \quad 0} \frac{u}{} \mathrm{Tr} \; Š \; \frac{}{r} D^{\mu \; F}_{sig,e}(r) \; \exp Š u \; D^{\mu \; F}_{sig,e}(r)^2$$
$$= \lim_{u \quad 0} \mathrm{Tr} \; Š \; \overline{Š1} \; \frac{u}{} D^{\mu \; F}_{sig,e} \exp Š u \; D^{\mu \; F}_{sig,e}(r)^2$$
$$= Š \lim_{u \quad 0} \frac{\overline{Š1}}{2} \mathrm{Tr} \; \bar{u} D^{\mu \; F} \exp Š u \; D^{\mu \; F}(r)^2 \; . \tag{3.85}$$

Now we do the rescaling as in Sect. 3.5, then by (3.23), we know the rescaled operator of $\sqrt{u}D^{\mu\otimes F}$ is

$$-\sum_{i=1}^{n} c(e_i)\left(\nabla_{e_i} + \frac{1}{4}\langle R^{TM}_{x_0} y, e_i\rangle\right) + \frac{1}{2}\omega(F,h^F)_{x_0}. \tag{3.86}$$

By (3.57), (3.60), the rescaled operator of $(D^{\mu\otimes F}(0))^2$ is $L_{x_0}(0)$, and

$$\left(\sum_{i=1}^{n} c(e_i)\left(\nabla_{e_i} + \frac{1}{4}\langle R^{TM}_{x_0} y, e_i\rangle\right)\right)\exp\left(-L_{x_0}(0)\right)(0,0) \tag{3.87}$$

is zero. Thus as in Sect. 3.5, by (3.59), we get

$$\begin{aligned}
&\lim_{u\to 0} \mathrm{Tr}_s\left[\sqrt{u}D^{\mu\otimes F}\exp\left(-u\left(D^{\mu\otimes F}(0)\right)^2\right)\right]\\
&= (-1)^{\frac{m(m-1)}{2}+p+m}2^m\int_M dv_M \int^B \left(\widehat{A}(TB)\exp\left(-L_{x_0}(0)\right)(0,0)\right)\\
&\quad\times \mathrm{Tr}|_\mu\left[\exp\left(-R^\mu\right)\right]\mathrm{Tr}|_F\left[\frac{1}{2}\omega(F,h^F)\exp\left(\frac{1}{4}\omega(F,h^F)^2\right)\right].
\end{aligned} \tag{3.88}$$

By applying [5, Proposition 3.13], the Berezin integral of $\int^B$ in (3.88) is the coefficient of $e^1\wedge\cdots\wedge e^m$ of (cf. [30, (1.49)–(1.51)]) the Berezin integral $\int^B$ of

$$\begin{aligned}
&\left(-\frac{1}{4}\right)^{\frac{m}{2}}\left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}}\mathrm{Tr}|_\mu\left[\exp\left(-R^\mu\right)\right]\mathrm{Tr}|_F\left[\frac{1}{2}\omega(F,h^F)\exp\left(\frac{1}{4}\omega(F,h^F)^2\right)\right]\\
&\cdot\det{}^{1/2}\left(\frac{R^{TM}/2}{\sinh R^{TM}/2}\right)\det{}^{1/2}\left(\cosh\left(\frac{R^{THM}}{2}\right)\right)\det{}^{1/2}\left(\frac{\sinh(R^{TZ}/2)}{R^{TZ}/2}\right)\\
&\cdot\exp\left(\frac{1}{4}\sum_{i,j}\langle R^{TZ}e_i,e_j\rangle\hat{e}^i\wedge\hat{e}^j\right).
\end{aligned}$$

Now as p is odd, if n is even, then

$$-\left(\sqrt{-1}\right)^{\frac{p(p+1)}{2}+1}(-1)^{\frac{(p+n)(p+n+1)}{2}+p} = \left(\sqrt{-1}\right)^{-\frac{p-1}{2}}(-1)^{\frac{n(n-1)}{2}}.$$

Thus by (3.84), (3.85), (3.88) at $r=0$, we get

$$\left.\frac{\partial\bar{\eta}\left(D^{\mu\otimes F}_{sig,e}(r)\right)}{\partial r}\right|_{r=0} = -\frac{1}{2}\int_B \widehat{L}(TB)\mathrm{ch}(\mu)\int_Z e(TZ)\sum_{j=0}\frac{1}{j!}c_{2j+1}(F). \tag{3.89}$$

When we take derivative on t for (3.54), by (2.50), (3.50) and (3.89), we find,

$$\int_B L(TB)\,\mathrm{ch}(\mu)\int_Z e(TZ)\sum_{j=0}\frac{1}{j!}\int_Z e(TZ)c_{2j+1}(F)$$

$$= \int_B L(TB)\,\mathrm{ch}(\mu)\sum_{j=0}\frac{1}{j!}\Big(c_{2j+1}(H(Z,F|_Z)) - \mathrm{rk}(F)c_{2j+1}(H(Z,\mathbf{C}|_Z))\Big). \tag{3.90}$$

We claim that for any $j \in \mathbf{N}$,

$$c_{2j+1}(H(Z,\mathbf{C}|_Z)) = 0, \tag{3.91}$$

a proof of which will be given shortly.

As $L(TB)\,\mathrm{ch}(\cdot) : K(B)\otimes\mathbf{R} \to H^{\mathrm{even}}(B,\mathbf{R})$ is an isomorphism, we get from (3.90) and (3.91) that,

$$\sum_{j=0}\frac{1}{j!}\int_Z e(TZ)c_{2j+1}(F) = \sum_{j=0}\frac{1}{j!}c_{2j+1}(H(Z,F|_Z)) \quad \text{in } H^{\mathrm{odd}}(B,\mathbf{R}), \tag{3.92}$$

which is equivalent to the Bismut–Lott formula (1.2) through a simple degree counting, in the case where B is orientable and of odd dimension.

Proof of (3.91) If $n = \dim Z$ is odd, then as in (3.89), from (2.50), (3.53) and (3.89), for $F = \mathbf{C}$, we get

$$\int_B L(TB)\,\mathrm{ch}(\mu)\sum_{j=0}\frac{1}{j!}c_{2j+1}(H(Z,\mathbf{C}|_Z)) = 0. \tag{3.93}$$

As $L(TB)\,\mathrm{ch}(\cdot) : K(B)\otimes\mathbf{R} \to H^{\mathrm{even}}(B,\mathbf{R})$ is an isomorphism, we get (3.91).

If $n = \dim Z$ is even, then by (3.90), for $F = o(TZ)$ and the isomorphism $L(TB)\,\mathrm{ch}(\cdot) : K(B)\otimes\mathbf{R} \to H^{\mathrm{even}}(B,\mathbf{R})$, as $c_{2j+1}(o(TZ)) = 0$, we get

$$c_{2j+1}(H(Z,o(TZ)|_Z)) = c_{2j+1}(H(Z,\mathbf{R}|_Z)). \tag{3.94}$$

By Poincaré duality, we have for any $i \in \mathbf{N}$,

$$H^i(Z,o(TZ)|_Z) = \big(H^{n-i}(Z,\mathbf{R}|_Z)\big)^*. \tag{3.95}$$

Thus from [12, Theorem 1.8], (3.94) and (3.95), we get

$$-c_{2j+1}(H(Z,\mathbf{R}|_Z)) = c_{2j+1}(H(Z,\mathbf{R}|_Z)). \tag{3.96}$$

Thus we have proved (3.91).

3.8 A proof of Theorem 1.1

First of all, if B is not orientable, then there is a double covering B B such that B is orientable. We pull-back the bration : M B to get a bration : M B . Thus we only need to prove Theorem 1.1 when B is orientable.

From now on, we assume B is orientable.

Next, if B is of even dimension, then we can apply the analysis before to the product bration Z M × S^1 B × S^1 to get the result.

From now on, we also assume that dim B is odd.

Combining with what was done in the last subsection, we get a new proof of Bismut–Lott formula (1.2).

It remains to prove (1.4).

By the same argument as in (2.46), we have, when mod Q,

$$(M/B, \mu, F, 0) = \int_B L(TB)\mathrm{ch}(\mu) \int_Z e(TZ)\mathrm{Re}\ CCS\ F, {}^F . \tag{3.97}$$

Thus from Theorem 3.9, (2.46), (3.97), and the argument as in the proof of (3.92), we get in $H^{odd}(B, R/Q)$,

$$\int_Z e(TZ)\mathrm{Re}\ CCS\ F, {}^F$$
$$= \sum_{i=0}^{n} (-1)^i \mathrm{Re}\ CCS\ H^i(Z, F|_Z), {}^{H^i(Z,F|_Z)}$$
$$- rk(F) \sum_{i=0}^{n} (-1)^i \mathrm{Re}\ CCS\ H^i(Z, C|_Z), {}^{H^i(Z,C|_Z)} . \tag{3.98}$$

Let $\overline{F}$ be the antidual bundle to F. Let : $\overline{F}$ × F C denote the pairing induced from the duality between $\overline{F}$ and F; it is linear in the rst factor and antilinear in the second factor. Let ${}^{\overline{F}}$ be the connection on $\overline{F}$ induced by ^{F}.

By Poincaré duality, one has for any nonnegative integer i

$$H^i(Z, (F\ o(TZ))|_Z) = H^{n-i}(Z, F|_Z) . \tag{3.99}$$

Thus, if $(F, {}^F)$ $(\overline{F}, {}^{\overline{F}})$, then $H^i(Z, F|_Z)$ $\overline{H^i(Z, F|_Z)}$ and one has

$$\sum_{i=0}^{n} (-1)^i \mathrm{Re}\ CCS\ H^i(Z, (F\ o(TZ))|_Z), {}^{H^i(Z,(F\ o(TZ))|_Z)}$$
$$= \sum_{i=0}^{n} (-1)^{n-i} \mathrm{Re}\ CCS\ H^i(Z, F|_Z) , {}^{(H^i(Z,F|_Z))}$$

$$= \sum_{i=0}^{n} (-1)^{n-i} \mathrm{Re}\left(CCS\left(\overline{H^i(Z, F|_Z)}, \nabla^{\overline{H^i(Z,F|_Z)}} \right) \right)$$

$$= (-1)^n \sum_{i=0}^{n} (-1)^i \mathrm{Re}\left(CCS\left(H^i(Z, F|_Z), \nabla^{H^i(Z,F|_Z)} \right) \right). \qquad (3.100)$$

If $n = \dim Z$ is odd, then by setting $F = \mathbf{C}_{o(TZ)}$ in (3.98) and by (3.100), one gets

$$2 \sum_{i=0}^{n} (-1)^i \mathrm{Re}\left(CCS\left(H^i(Z, \mathbf{C}|_Z), \nabla^{H^i(Z,\mathbf{C}|_Z)} \right) \right) = 0 \ \text{ in } \ H^{\mathrm{odd}}(B, \mathbf{R}/\mathbf{Q}),$$
$$(3.101)$$

from which (1.6) follows.

On the other hand, if n is even, then (1.6) follows from the second part of [9, Theorem 3.12].

Thus (1.6) holds in its full generality.

From (1.6) and (3.98), one gets (1.4).

The proof of Theorem 1.1 is completed.

3.9 A refinement in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$: Proof of Theorem 1.2

Recall that $\pi : M \to B$ is a fibration of compact smooth manifolds with compact fiber Z, and g^{TZ} is a metric on TZ. Let (F, ∇^F) be a complex flat vector bundle and h^F is a Hermitian metric on F. Recall also that $\nabla^{F,e}$ is the Hermitian connection on F induced by ∇^F and h^F as in (2.7).

Suppose that Z is even dimensional and spin^c. Let $S(TZ) = S^+(TZ) \oplus S^-(TZ)$ be the spinor bundle of TZ.

In [25, Sect. 4], Lott defined a topological index $\mathrm{Ind}_{\mathrm{top}}$ and an analytic index $\mathrm{Ind}_{\mathrm{an}}$, mapping from $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ to $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$ (cf. Sect. 2.6).

Especially, for any $\mathcal{E} \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ [25, (37)],

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{Ind}_{\mathrm{top}}(\mathcal{E}) \right) = \int_Z \widehat{A}(TZ) e^{c_1(L_Z)/2} \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}) \in H^*(B, \mathbf{R}/\mathbf{Q}), \qquad (3.102)$$

where $\widehat{A}(TZ, \nabla^{TZ}) = \det{}^{1/2}\left(\frac{R^{TZ}/2}{\sinh(R^{TZ}/2)} \right)$ and the cohomology class of $\widehat{A}(TZ, \nabla^{TZ})$ is the A-hat class $\widehat{A}(TZ)$ of TZ, while $c_1(L_Z)$ is the first Chern class of the complex line bundle L_Z which defines the spin^c structure of TZ.

The main result of Lott [25, Corollaries 1 and 3] is that for any $\mathcal{E} \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$,

$$\mathrm{Ind}_{\mathrm{top}}(\mathcal{E}) = \mathrm{Ind}_{\mathrm{an}}(\mathcal{E}). \qquad (3.103)$$

We denote by C the trivial complex line bundle carrying the trivial metric and connection. Then $\mathcal{F} = [(F, h^F, \ {}^{F,e}, 0) - \mathrm{rk}(F)C] \ K^{-1}_{R/Z}(M)$, thus $(S^+(TZ) - S^-(TZ)) \ \mathcal{F} \ K^{-1}_{R/Z}(M)$.

Recall that $I(F)$ was de ned in (1.7).

Let $(\ {}^F, h^F) \ {}^{\mathrm{odd}}(B)/\mathrm{Im}(d)$ be the eta form of Bismut–Cheeger [10, De nition 4.33] de ned by

$$ {}^F, h^F = 2 \ \overline{-1}^{\ -\frac{1}{2}} \ _0 \ \mathrm{Tr}_s \ \frac{C_u}{u} \exp(-C_u^2) \ du. \tag{3.104}$$

We may view it as an element in $K^{-1}_{R/Z}(B)$ represented by $(0, 0, 0, (\ {}^F, h^F))$.

Theorem 3.14

$$\begin{aligned} &\mathrm{Ind}_{top} \ S^+(TZ) - S^-(TZ) \ \ \mathcal{F} \\ &= I(F) - \ {}^F, h^F - \mathrm{rk}(F) \ I(C) - \ {}^C, h^C \ . \end{aligned} \tag{3.105}$$

Proof In [25], Lott used spin^c Dirac operator to de ne Ind_{an}, especially, the spin^c Dirac operator (twisted by $(S^+(TZ) - S^-(TZ)) \ F)$ $D^{Z,c}$ is

$$D^{Z,c} = \ _j \ c(e_j) \ {}^{TZ}_{e_j} \ {}^{F,e} = D^Z + \frac{1}{2} \ _j \ c(e_j) \ \ F, h^F \ (e_j). \tag{3.106}$$

The operator $D^{Z,c}$ does not have berwise constant dimensional kernels, thus we need to choose smooth nite dimensional sub-bundles $F_\pm$ of $E_\pm$ ($E_+ = E^{\mathrm{even}}$, $E_- = E^{\mathrm{odd}}$) and complementary subbundles $G_\pm$ such that $D^{Z,c}$ are diagonal with respect to the decompositions $E_\pm = F_\pm \ G_\pm$ and $D^{Z,c}$ restricted to $G_\pm$ is invertible (cf. [25, De nition 14]).

It seems that it is hard to compare directly the right hand side of (3.105) to $\mathrm{Ind}_{an}((S^+(TZ) - S^-(TZ)) \ \mathcal{F})$ in [25, De nition 14]. But by the arguments in [25, Proposition 6 and Corollary 1], we will get (3.105) if we can prove the following identity for any odd dimensional compact spin^c manifold B,

$$\bar{\ }_M \ S^+(TZ) - S^-(TZ) \ \ F = \bar{\ }_B \ I(F) - \ {}^F, h^F \quad \text{in } R/Z, \tag{3.107}$$

where $\bar{\ }$ is the reduced eta invariant of the spin^c Dirac operator twisted by the corresponding bundles as in [25, De nition 11], i.e., for the tuples $(\mathcal{G}, h^{\mathcal{G}}, \ {}^{\mathcal{G}}, \ )$ on B as in Sect. 2.6, let $D^{\mathcal{G}^\pm}$ be the spin^c Dirac operator on B twisted by the bundle $\mathcal{G}^\pm$ and its reduced eta invariant $\bar{\eta}(D^{\mathcal{G}^{\pm}})$ as in (2.24), then

$$\bar{\eta}_B(\mathcal{G}, h^{\mathcal{G}}, \nabla^{\mathcal{G}}, \ ) = \bar{\eta}(D^{\mathcal{G}^{+}}) - \bar{\eta}(D^{\mathcal{G}^{-}}) - \int_B \widehat{A}(TB, \nabla^{TB}) e^{c_1(L_B)/2} \quad \text{in } \mathbb{R}/\mathbb{Z}, \tag{3.108}$$

with $c_1(\nabla^{L_B})$ is the first Chern form of the complex line bundle L_B which defines the spin^c structure of TB, and ∇^{TB} is the Levi-Civita connection on (TB, g^{TB}).

Let $\nabla^{S(TB)}$ be the connection on the spinor bundle $S(TB)$ induced by ∇^{TB} and the connection on the line bundle defining the spin^c structure. Let ∇^{e} be the connection on $\Lambda(T^*Z) \otimes \pi^*(S(TB)) \otimes F$ induced by $\nabla^{\Lambda(T^*Z)}$, $\nabla^{\pi^*(S(TB))}$, $\nabla^{F,e}$.

Set

$$D^H = \sum c(f_\alpha) \nabla^{e}_{f_\alpha} + \frac{1}{2}k(f_\alpha) . \tag{3.109}$$

By proceeding as in [10, (4.26)], we get the spin^c Dirac operator $D^{M,c}$ (twisted by $(S^+(TZ) - S^-(TZ))^* \otimes F$), on M is

$$D^{M,c} = D^H + D^{Z,c} - \frac{c(T)}{4}. \tag{3.110}$$

and $\bar{\eta}_M((S^+(TZ) - S^-(TZ))^* \otimes F)$ is the reduced eta invariant of the operator $D^{M,c}$.

Let

$$D^M = D^H + D^Z - \frac{c(T)}{4}. \tag{3.111}$$

Then $D^M = D^{M,c} - \frac{1}{2}\sum_j c(e_j)\,\omega(F, h^F)(e_j)$ and $\frac{1}{2}\sum_j c(e_j)\,\omega(F, h^F)(e_j)$ anti-commutes with $c(X)$, $X \in TM$.

Using the variation formula for eta invariants (cf. [3] and [11, Proposition 2.8]) and the local index techniques as in [10, Theorem 2.7], we know (cf. [9, Proposition 3.5] and [30, (3.5)])

$$\bar{\eta}_M((S^+(TZ) - S^-(TZ))^* \otimes F) = \bar{\eta}(D^{M,c}) = \bar{\eta}(D^M) \quad \bmod \mathbb{Z}. \tag{3.112}$$

From (3.112), we can use the adiabatic limit argument as in [25, Proposition 6] to get (3.107).

By [9, Theorem 3.7] and [30, Proposition 2.3] (cf. also Lemma 3.13),

$$\ \ (F, h^F) = 0. \tag{3.113}$$

Thus from (3.104) and (3.105), we get (1.8). Thus the proof of Theorem 1.2 is complete.

When we apply (3.102) to (1.8), we get again (3.98). Thus (1.8) represents a re - nement of (3.98) in $K^{-1}_{R/Z}(B)$.

We leave the interested reader to extend this to the case where no spin assumption on TZ is required.

3.10 The and torsion forms

In the rest of this section, the supertrace Tr_s on E is de ned by the Z_2-grading induced from $(-1)^{N_Z}$.

Recall $n = \dim Z$. Let

$$d(H(Z,F)) = \sum_{i=0}^{n} (-1)^i \, i \dim H^i(Z,F). \qquad (3.114)$$

From (3.84), we denote

$$a_{-1} = (-1)^{\frac{n(n+1)}{2}} \quad -\tfrac{n}{2}\mathrm{rk}(F) \int_Z \quad {}^B \frac{1}{2} \sum_{i=1}^{n} e^i \quad e^i \exp \left(-\frac{1}{2} R^{TZ}\right) . \qquad (3.115)$$

Then a_{-1} is a function on B and is 0 if n is even.

For any $u > 0$, let $\ _u : (\ T\ B) \quad (\ T\ B)$ be de ned by that for $\ (\ T\ B)$, $\ _u\ = u^{-\deg\ /2}\ $.

By (3.14), one nds

$$\mathrm{Tr}_s \ N_Z \exp \quad 1 + r^2 \quad D_u^2 \quad = \ _u \mathrm{Tr}_s \ N_Z \exp \quad 1 + r^2 \quad u D_1^2 \quad . \qquad (3.116)$$

By standard results on heat kernels (cf. [5, Theorem 2.30]), we know that $\mathrm{Tr}_s[N_Z \exp((1+r^2)uD_1^2)]$ has an asymptotic expansion in u as $u \quad 0^+$, which only contains integral powers of u if $n = \dim Z$ is even, and only contains half-integral powers of u if n is odd. Since $\mathrm{Tr}_s[N_Z \exp((1+r^2)D_u^2)]$ is an even form on B, by (3.116) we see that the same happens to it.

On the other hand, as in [13, (11.1)], we have

$$N_Z = \frac{1}{2} \sum_{i=1}^{n} c(e_i)c(e_i) + \frac{n}{2}. \qquad (3.117)$$

By the proof of [12, Theorem 3.15], (3.14) and (3.117), as in [12, Theorem 3.21] and [13, Theorem 7.10], we have for $\ \ R$, as $u \quad 0^+$,

$$\mathrm{Tr}_s \ N_Z \exp \quad 1 + r^2 \quad D_u^2 \quad = \begin{cases} \frac{n}{2} \ (\ Z)\mathrm{rk}(F) + O(u) & \text{if } n \text{ is even}, \\ \frac{2a_{-1}}{\overline{1 + r^2 \quad u}} + O\left(\sqrt{u}\right) & \text{if } n \text{ is odd} \end{cases} \qquad (3.118)$$

While as u + ,

$$\mathrm{Tr}_s \; N_Z \exp \; 1 + r^2 \; D_u^2 \;=\; d(H(Z,F)) + O\left(\frac{1}{\sqrt{u}}\right). \tag{3.119}$$

From (3.79) and (3.118), we know that as u 0^+,

$$\mathrm{Tr}_s \; \frac{}{u} \; C_u + \sqrt{-1}\, r D_u \; \exp -\; C_u + \sqrt{-1}\, r D_u \;^2$$

$$= \frac{\sqrt{-1}\, r\, d a_{-1}}{\sqrt{1+r^2}\, u^{3/2}} + O\left(\frac{1}{\sqrt{u}}\right). \tag{3.120}$$

The following de nition is closely related to [12, De nition 3.22].

DeÞnition 3.15 For any r R, put

$$I_r = -r \int_0^{+} \mathrm{Tr}_s \; N_Z e^{-(C_u + \sqrt{-1} r D_u)^2} - d(H(Z, F|_Z)) - + \frac{2a_{-1}}{(1+r^2)u}$$

$$- \frac{n}{2} \;(\;Z) \mathrm{rk}(F) - d(H(Z, F|_Z)) \; e^{-(1+r^2)u/4} \; \frac{du}{2u}. \tag{3.121}$$

Let $_r$ be the -form of Bismut–Cheeger [10, De nition 4.33] de ned by

$$_r = 2\; \sqrt{-1}^{-\frac{1}{2}} \int_0 \; \mathrm{Tr}_s \; \frac{}{u} \; C_u + \sqrt{-1}\, r D_u$$

$$\times \exp -\; C_u + \sqrt{-1}\, r D_u \;^2 \;-\; \frac{\sqrt{-1}\, r\, d a_{-1}}{\sqrt{1+r^2}\, u^{3/2}} \; du. \tag{3.122}$$

Remark 3.16 The extra term involving $d a_{-1}$ in the right hand side of (3.122) shows that this $_r$ form is slightly different from what in [10].

Theorem 3.17 *For any r R, the form $_r$ is exact and is 0 at $r = 0$. Moreover, the following transgression formula holds,*

$$_r = -\frac{1}{2} d I_r. \tag{3.123}$$

Proof Theorem 3.17 is a direct consequence of Lemma 3.13, (3.118) and (3.119).

Let $T_f(T^H M, g^{TZ}, h^F)$ be the torsion form constructed in the spirit of [12, De - nition 3.21] associated to the odd holomorphic function $f(z)$ such that $f\,(z) = e^{z^2}$, that is

$$\mathcal{T}_f\left(T^HM, g^{TZ}, h^F\right) = -\int_0^{+\infty} \Big\{ \mathrm{Tr}_s\left[N_Z e^{D_u^2}\right] - d(H(Z, F|_Z)) - \frac{2a_{-1}}{\sqrt{u}} - \left(\frac{n}{2}\chi(Z)\mathrm{rk}(F) - d(H(Z, F|_Z))\right) e^{-u/4} \Big\} \frac{du}{2u}. \tag{3.124}$$

Theorem 3.18 The following identity holds,

$$\frac{\partial I_r}{\partial r}\Big|_{r=0} = \mathcal{T}_f\left(T^HM, g^{TZ}, h^F\right). \tag{3.125}$$

In particular,

$$\frac{\partial\ _r}{\partial r}\Big|_{r=0} = -\frac{1}{2} d\mathcal{T}_f\left(T^HM, g^{TZ}, h^F\right). \tag{3.126}$$

Proof Formula (3.125) follows from (3.121), (3.124). Formula (3.126) follows from (3.123) and (3.125).

Theorem 3.19 For any $r \in \mathbb{R}$, the following identity holds,

$$\begin{aligned} {}_r = \frac{r}{2} \sum_{j=0}^{+\infty} \frac{\left(1+r^2\right)^j}{j!(2j+1)} \sum_{i=0}^{n} (-1)^i c_{2j+1}\left(H^i(Z, F|_Z), h^{H^i(Z,F|_Z)}\right) \\ - \frac{r}{2} \sum_{j=0}^{+\infty} \frac{\left(1+r^2\right)^j}{j!(2j+1)} \int_Z e\left(TZ, \nabla^{TZ}\right) c_{2j+1}\left(F, h^F\right). \end{aligned} \tag{3.127}$$

In particular,

$$\begin{aligned} \frac{\partial\ _r}{\partial r}\Big|_{r=0} = \frac{1}{2} \sum_{j=0}^{+\infty} \frac{1}{j!(2j+1)} \sum_{i=0}^{n} (-1)^i c_{2j+1}\left(H^i(Z, F|_Z), h^{H^i(Z,F|_Z)}\right) \\ - \frac{1}{2} \sum_{j=0}^{+\infty} \frac{1}{j!(2j+1)} \int_Z e\left(TZ, \nabla^{TZ}\right) c_{2j+1}\left(F, h^F\right). \end{aligned} \tag{3.128}$$

Proof In fact, by Lemma 3.13 for $r = 0$ and (3.122), we have

$$ {}_r = \sqrt{-1}\, r \left(2\sqrt{-1}\right)^{-\frac{1}{2}} \int_0^{+\infty} \left\{ \mathrm{Tr}_s\left[\frac{D_u}{u} \exp\left(\left(1+r^2\right) D_u^2\right)\right] - \frac{da_{-1}}{\left(1+r^2\right)u^{3/2}} \right\} du. \tag{3.129}$$

From (3.14) and (3.129), one deduces that

$$ {}_r = \frac{r}{\sqrt{1+r^2}} \, (1+r^2)^{-1} \frac{\partial}{\partial r}\Big|_{r=0} . \tag{3.130}$$

By (3.130), we need only to prove (3.128).

Lemma 3.20 The following identity holds,

$$\mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u}\exp\left(-D_u^2\right)\right] - \frac{d a_{-1}}{u^{3/2}} = \frac{\partial}{\partial u}\int_0^1 \mathrm{Tr}_s\left[D_u \exp\left(-s^2 D_u^2\right)\right] ds. \tag{3.131}$$

Proof By (3.14), (3.79) and (3.118), one has

$$\lim_{s\to 0^+} \mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u} s\exp\left(-s^2D_u^2\right)\right] = \lim_{s\to 0^+} \frac{s}{2u} d\, {}_{s^{-2}}\mathrm{Tr}_s\left[N_Z \exp\left(-D^2_{s^2u}\right)\right] = \frac{d a_{-1}}{u^{3/2}}. \tag{3.132}$$

Thus

$$\begin{aligned}
&\frac{\partial}{\partial u}\int_0^1 \mathrm{Tr}_s[D_u \exp\left(-s^2D_u^2\right)]\, ds \\
&= \int_0^1 \mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u}\exp\left(-s^2D_u^2\right)\right] ds + \int_0^1 \mathrm{Tr}_s\left[D_u\left(-s^2\left[D_u, \frac{\partial D_u}{\partial u}\right]\right)\exp\left(-s^2D_u^2\right)\right] ds \\
&= \int_0^1 \mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u}\left(1 + 2s^2D_u^2\right)\exp\left(-s^2D_u^2\right)\right] ds \\
&= \mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u}\int_0^1 \frac{\partial}{\partial s}\left(s\exp\left(-s^2D_u^2\right)\right) ds\right] = \mathrm{Tr}_s\left[\frac{\partial D_u}{\partial u}\exp\left(-D_u^2\right)\right] - \frac{d a_{-1}}{u^{3/2}},
\end{aligned} \tag{3.133}$$

which is exactly (3.131).

Now by (3.14) again, one has

$$\mathrm{Tr}_s\left[D_u \exp\left(-s^2D_u^2\right)\right] = \frac{1}{s}\, {}_{s^{-2}}\mathrm{Tr}_s\left[D_{s^2u}\exp\left(-D^2_{s^2u}\right)\right]. \tag{3.134}$$

From [12, Theorem 3.16] and (3.83), we know that

$$(2\pi\sqrt{-1})^{\frac{1}{2}}\varphi\,\mathrm{Tr}_s\left[D_u\exp\left(-D_u^2\right)\right] = \begin{cases} \int_Z e(TZ,\nabla^{TZ})\sum_{j=0}^{\infty}\frac{1}{j!}c_{2j+1}\left(F,h^F\right)+O\left(\sqrt{u}\right) & \text{as } u\to 0, \\ \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(H^i(Z,F|_Z),h^{H^i(Z,F|_Z)}\right)+O\left(\frac{1}{\sqrt{u}}\right) & \text{as } u\to+\infty. \end{cases} \tag{3.135}$$

Since $\varphi\mathrm{Tr}_s[D_u\exp(s^2D_u^2)]$ is an odd form on B, by (3.135), one sees that $|\frac{1}{s}\varphi s^{-2}\mathrm{Tr}_s[D_{s^2u}\exp(D_{s^2u}^2)]|$ has a fixed uniform upper bound for $s\in(0,1]$, $u\in(0,+\infty)$.

Thus, from (3.129), (3.131), (3.134), (3.135) and the dominated convergence property, we get

$$\begin{aligned} 2\frac{\partial\theta}{\partial r}\Big|_{r=0} &= (2\pi\sqrt{-1})^{1/2}\lim_{u\to+\infty}\int_0^1\frac{1}{s}\varphi s^{-2}\mathrm{Tr}_s\left[D_{s^2u}\exp\left(D_{s^2u}^2\right)\right]ds \\ &\quad -\lim_{u\to 0^+}\int_0^1\frac{1}{s}\varphi s^{-2}\mathrm{Tr}_s\left[D_{s^2u}\exp\left(D_{s^2u}^2\right)\right]ds \\ &= \sum_{j=0}^{\infty}\frac{1}{j!}\int_0^1 s^{2j}ds\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(H^i(Z,F|_Z),h^{H^i(Z,F|_Z)}\right) \\ &\quad -\int_Z e(TZ,\nabla^{TZ})\sum_{j=0}^{\infty}\frac{1}{j!}c_{2j+1}\left(F,h^F\right)\int_0^1 s^{2j}ds, \end{aligned} \tag{3.136}$$

which is equivalent to (3.128).

Combining (3.126) and (3.128), one gets the following transgression formula of Bismut–Lott type.

Corollary 3.21 The following identity holds,

$$\begin{aligned} d\mathcal{T}_f\left(T^HM,g^{TZ},h^F\right) &= \sum_{j=0}^{+\infty}\frac{1}{j!(2j+1)}\int_Z e\left(TZ,\nabla^{TZ}\right)c_{2j+1}\left(F,h^F\right) \\ &\quad -\sum_{j=0}^{+\infty}\frac{1}{j!(2j+1)}\sum_{i=0}^{n}(-1)^i c_{2j+1}\left(H^i(Z,F|_Z),h^{H^i(Z,F|_Z)}\right). \end{aligned} \tag{3.137}$$

Remark 3.22 Formula (3.137) is equivalent to (1.3) via the following more precise relation between $\mathcal{T}_f(T^HM, g^{TZ}, h^F)$ and the Bismut–Lott torsion form $\mathcal{T}(T^HM, g^{TZ}, h^F)$ defined in [12, Definition 3.22].

Theorem 3.23 *The following identity holds in (B),*

$$\mathcal{T}\left(T^HM, g^{TZ}, h^F\right) = (1+N_B)\,\mathcal{T}_f\left(T^HM, g^{TZ}, h^F\right). \tag{3.138}$$

Proof Recall that the Bismut–Lott torsion form $\mathcal{T}(T^HM, g^{TZ}, h^F)$ is defined by

$$\begin{aligned}\mathcal{T}\left(T^HM, g^{TZ}, h^F\right) = -\int_0^{+\infty}\Big[&\varphi\,\mathrm{Tr}_s\left[N_Z\left(1+2D_u^2\right)e^{D_u^2}\right] - d(H(Z, F|_Z))\\ &-\left(\frac{n}{2}\chi(Z)\mathrm{rk}(F) - d(H(Z, F|_Z))\right)\left(1-\frac{u}{2}\right)e^{-u/4}\Big]\frac{du}{2u}.\end{aligned} \tag{3.139}$$

A direct computation shows that the 0-form component of $\mathcal{T}_f(T^HM, g^{TZ}, h^F)$ is exactly the half of the Ray–Singer analytic torsion defined in [28] and [13]. Thus, the 0-form component of (3.138) is a consequence of [12, Theorem 3.29].

On the other hand, for $i > 0$, we denote by a superscript $[i]$ the i-form component of the corresponding forms. Then by (3.14), one has

$$\left\{\varphi\,\mathrm{Tr}_s\left[N_Z D_u^2 \exp\left(D_u^2\right)\right]\right\}^{[i]} = u^{-i/2}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z u D_1^2 \exp\left(uD_1^2\right)\right]\right\}^{[i]}. \tag{3.140}$$

Thus, one deduces that

$$\begin{aligned}&\int_0^{+\infty}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z D_u^2 \exp\left(D_u^2\right)\right]\right\}^{[i]}\frac{du}{u}\\ &= \int_0^{+\infty} u^{-\frac{i}{2}}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z D_1^2 \exp\left(uD_1^2\right)\right]\right\}^{[i]} du\\ &= \int_0^{+\infty} u^{-\frac{i}{2}}\frac{\partial}{\partial u}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z \exp\left(uD_1^2\right)\right]\right\}^{[i]} du\\ &= \int_0^{+\infty} i\,u^{-\frac{i}{2}}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z \exp\left(uD_1^2\right)\right]\right\}^{[i]}\frac{du}{2u},\end{aligned} \tag{3.141}$$

where in the last equality we have used the facts that

$$\lim_{u\to 0^+} u^{-\frac{i}{2}}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z \exp\left(uD_1^2\right)\right]\right\}^{[i]} = \lim_{u\to 0^+}\left\{\varphi\,\mathrm{Tr}_s\left[N_Z \exp\left(D_u^2\right)\right]\right\}^{[i]} = 0, \tag{3.142}$$

and

$$\lim_{u\to+\infty} u^{-\frac{i}{2}} \left\{ \mathrm{Tr}_s\left[N_Z \exp\left(-uD_1^2\right)\right]\right\}^{[i]} = \lim_{u\to+\infty} \left\{ \mathrm{Tr}_s\left[N_Z \exp\left(-D_u^2\right)\right]\right\}^{[i]} = 0, \tag{3.143}$$

which are consequences of (3.118) and (3.119).

From (3.141), we get (3.138).

Remark 3.24 From (3.79), as in (2.74), (2.75), we get

$$\begin{aligned} &\frac{\partial}{\partial r}\left\{\frac{r}{2u} d\mathrm{Tr}_s\left[N_Z \exp\left(-\left(C_u + \sqrt{-1}\, r D_u\right)^2\right)\right]\right\} \\ &\quad - \frac{\partial}{\partial u}\mathrm{Tr}_s\left[D_u \exp\left(-\left(C_u + \sqrt{-1}\, r D_u\right)^2\right)\right] \\ &= \frac{-1}{2u} d\mathrm{Tr}_s\left[2N_Z D_u^2 \exp\left(-\left(C_u + \sqrt{-1}\, r D_u\right)^2\right)\right]. \end{aligned} \tag{3.144}$$

Especially, when we restrict ourselves to $r = 0$, from (3.144), we get

$$\frac{\partial}{\partial u}\mathrm{Tr}_s\left[D_u \exp\left(-C_u^2\right)\right] = \frac{1}{2u} d\mathrm{Tr}_s\left[N_Z\left(1 + 2D_u^2\right)\exp\left(-C_u^2\right)\right]. \tag{3.145}$$

This is exactly [12, Theorem 3.20]. It seems interesting that here we obtain it purely through the consideration of forms.

Acknowledgments We would like to thank Professor Jean-Michel Bismut for kindly informing us his results in [9], when we told him our results related to (1.4). We also thank the referee for his useful comments and suggestions. This work was supported in part by MOEC, the 973 project of MOSTC and NNSFC. The authors would like to thank Huitao Feng for helpful discussions. Part of the work was done while the second author was visiting the Center of Mathematical Sciences at Zhejiang University in December of 2003, he would like to thank Kefeng Liu and Hongwei Xu for hospitality.

References

1. Atiyah, M.F., Hirzebruch, F.: Riemann-Roch theorems for differentiable manifolds. Bull. Am. Math. Soc. 65, 276–281 (1959)
2. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry I. Proc. Camb. Philos. Soc. 77, 43–69 (1975)
3. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry II. Proc. Camb. Philos. Soc. 78, 405–432 (1975)
4. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry III. Proc. Camb. Philos. Soc. 79, 71–99 (1976)
5. Berline, N., Getzler, E., Vergne, M.: Heat kernels and the Dirac operator. Grundl, Math. Wiss. 298. Springer, Berlin (1992)
6. Berthomieu, A., Bismut, J.-M.: Quillen metric and higher analytic torsion forms. J. Reine Angew. Math. 457, 85–184 (1994)
7. Bismut, J.-M.: The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs. Invent. Math. 83, 91–151 (1986)

8. Bismut, J.-M.: Local index theory, eta invariants and holomorphic torsion: a survey. Surveys in Differential Geometry, Vol. III, International Press, Boston, MA, pp 1–76 (1998)
9. Bismut, J.-M.: Eta invariants, differential characters and at vector bundles. With an Appendix by K. Corlette and H. Esnault. Preprint, 1995. Chinese Ann. Math. Ser. B 26(1), 15–44 (2005)
10. Bismut, J.-M., Cheeger, J.: -invariants and their adiabatic limits. J. Am. Math. Soc. 2, 33–70 (1989)
11. Bismut, J.-M., Freed, D.S.: The analysis of elliptic families, II. Commun. Math. Phys. 107, 103–163 (1986)
12. Bismut, J.-M., Lott, J.: Flat vector bundles, direct images and higher real analytic torsion. J. Am. Math. Soc. 8, 291–363 (1995)
13. Bismut, J.-M., Zhang, W.: An extension of a theorem by Cheeger and Müller. Astérisque, tom. 205, Paris (1992)
14. Bloch, S., Esnault, H.: Algebraic Chern-Simons theory. Am. J. Math. 119(4), 903–952 (1997)
15. Bloch, S., Esnault, H.: A Riemann-Roch theorem for at bundles, with values in the algebraic Chern-Simons theory. Ann. Math. (2) 151(3), 1025–1070 (2000)
16. Bunke, U.: Index theory, eta forms, and Deligne cohomology. Preprint, math.DG/0201112
17. Bunke, U., Ma, X.: Index and secondary index theory for at bundles with duality. In: Gil Juan et al. (Eds.) Aspects of Boundary Problems in analysis and geometry. A volume of advances in partial differential equations. Birkhäuser, Basel. Oper. Theory, Adv. Appl. 151, 265–341 (2004)
18. Bunke, U., Schick, T.: Smooth K-theory. Preprint, arXiv:0707.0046
19. Cheeger, J., Simons, J.: Differential characters and geometric invariants. in Lecture Notes in Math. Vol. 1167, pp. 50–80. Springer, Berlin (1985)
20. Dai, X.: Adiabatic limits, non multiplicativity of signature and Leray spectral sequence. J. Am. Math. Soc. 4, 265–321 (1991)
21. Dwyer, W., Weiss, M., Williams, B.: A parametrized index theorem for the algebraic K-theory Euler class. Acta Math. 190, 1–104 (2003)
22. Esnault, H.: Characteristic classes of at bundles. II. K-Theory 6(1), 45–56 (1992)
23. Esnault, H.: Algebraic differential characters. Regulators in Analysis, Geometry and Number Theory, pp. 89–115, Progress in Mathematics, 171, Birkhäuser, Boston, MA (2000)
24. Esnault, H.: Characteristic classes of at bundles and determinant of the Gauss-Manin connection. Proc. ICM2002, Vol. II, pp. 471–481, Higher Education Press, Beijing (2002)
25. Lott, J.: R/Z-index theory. Commun. Anal. Geom. 2, 279–311 (1994)
26. Ma, X.: Functoriality of real analytic torsion forms. Israel J. Math. 131, 1–50 (2002)
27. Quillen, D.: Superconnections and the Chern character. Topology 24, 89–95 (1986)
28. Ray, D.B., Singer, I.M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. 7, 145–210 (1971)
29. Soulé, C.: Classes caractéristiques secondaires des brés plats. Séminaire Bourbaki, Vol. 1995/96. Astérisque No. 241 (1997), Exp. No. 819, 5, 411–424 (1997)
30. Zhang, W.: Sub-signature operators, -invariants and a Riemann-Roch theorem for at vector bundles. Chin. Ann. Math. 25B, 7–36 (2004)
31. Zhang, W.: Sub-signature operators and a local index theorem for them (in Chinese). Chin. Sci. Bull. 41, 294–295 (1996)