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# A criterion of decomposability for degree 4 algebras with unitary involution

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## Abstract

Let  $A$  be a degree 4 central simple algebra endowed with a unitary involution  $\sigma$ . We prove that  $(A, \sigma)$  is decomposable if and only if its discriminant (i.e. the Brauer class of its discriminant algebra) is trivial. © 2000 Elsevier Science B.V. All rights reserved.

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## 0. Introduction

Let  $K$  be an arbitrary field. A central simple  $K$ -algebra  $A$  endowed with an involution  $\sigma$  (not necessarily trivial on  $K$ ) is said to be decomposable if  $A$  is tensor product over  $K$  of two proper subalgebras stable under  $\sigma$ .

The study of the decomposability of algebras with involution is a classical question (see e.g. [3, 12]). Let us assume that the algebra  $A$  is of degree 4. Rowen [11, Theorem B] proved that in the symplectic case  $(A, \sigma)$  is always decomposable. In the orthogonal case, Knus et al. [7, Theorem 3.1] (see also [6, Corollary 5.4]) gave a

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criterion of decomposability using the classical notion of *discriminant* of an orthogonal involution (also called *pfaffian discriminant*):  $(A, \sigma)$  is decomposable if and only if the discriminant of  $\sigma$  is trivial.

The purpose of this paper is to give an analogous criterion in the unitary case using the *discriminant algebra* of  $(A, \sigma)$  introduced in [5] (see Theorem 3.1). When the base field is of characteristic different from 2, the criterion can also be formulated in terms of the *determinant class modulo 2*, an invariant of unitary involutions introduced in [10] (see Remark 3.2).

## 1. Discriminant algebra, discriminant and determinant class modulo 2

Let  $A$  be a central simple  $K$ -algebra endowed with an involution  $\sigma$ . The involution is said to be of the first kind if it is trivial on  $K$  and of the second kind, or of unitary type, otherwise. We refer the reader to [13, Chap. 8] or [5, Chap. 1] for more detailed information on involutions.

To deal with the characteristic 2 case, we need to replace orthogonal involutions by quadratic pairs. One will find in [5, Section 5] an explanation of this fact and the definition of a quadratic pair. In characteristic  $\neq 2$ , this notion is equivalent to the notion of orthogonal involution.

We assume in this section that  $A$  is of even degree  $n = 2m$  and that  $\sigma$  is of unitary type. We denote by  $k$  the subfield of  $K$  fixed by  $\sigma$ . Thus  $K/k$  is a separable quadratic extension.

In [5, Section 10], Knus et al. associate to  $(A, \sigma)$  a degree  $\binom{n}{m}$  central simple  $k$ -algebra  $\mathcal{D}(A, \sigma)$  called the *discriminant algebra* of  $(A, \sigma)$ . If  $m$  is even, then the discriminant algebra is endowed with a quadratic pair.

The Brauer class  $[\mathcal{D}(A, \sigma)]$  of the discriminant algebra will be called the *discriminant* of  $(A, \sigma)$ . This terminology is motivated by the following remark. If the exponent of  $A$  divides  $m$ , then the algebra  $\mathcal{D}(A, \sigma) \otimes_k K$  is split. Indeed, it is isomorphic by [5, Proposition 10.30] to  $\lambda^m A$ , which is Brauer equivalent to  $A^{\otimes m}$ . Hence, in this case, the discriminant of  $(A, \sigma)$  is an element of the relative Brauer group  $\text{Br}(K/k)$  which is canonically isomorphic to the quotient  $k^*/N_{K/k}(K^*)$ , where  $N_{K/k}$  is the norm homomorphism.

If the base field has characteristic different from 2, the second named author attached to  $(A, \sigma)$  an invariant called *determinant class modulo 2*, which can be defined either using Galois cohomology, or via the trace forms of  $(A, \sigma)$  (see [10] or [9]). This invariant, denoted by  $D(A, \sigma)$ , takes values in the 2-part  $\text{Br}_2(k)$  of the Brauer group of  $k$ .

It follows from [5, Section 11] that (in the characteristic  $\neq 2$  case) the Brauer class of  $\mathcal{D}(A, \sigma)$  is  $D(A, \sigma) + m[(-1, \alpha)]$ , where  $\alpha$  is an element of  $k^*$  such that  $K = k(\sqrt{\alpha})$  and where  $[(-1, \alpha)]$  is the Brauer class of the quaternion  $k$ -algebra  $(-1, \alpha)$ . In particular, if the degree of  $A$  is divisible by 4, then the determinant class modulo 2 coincides with the discriminant of  $(A, \sigma)$ .

## 2. Clifford algebra

Let  $k$  be a field (of arbitrary characteristic) and let  $(V, q)$  be a non-degenerate quadratic space over  $k$  (see [4, Definition 1.5 of Chap. I] or [13, Definition 6.9 of Chap. 1]). We recall that the *Clifford algebra*  $C(q)$  of the quadratic space  $(V, q)$  is by definition the tensor algebra of the vector space  $V$  modulo the two-sided ideal generated by the products  $v \otimes v - q(v)$ , where  $v$  ranges over  $V$ . For the properties of the Clifford algebra listed below we refer to [13, Chap. 9], [4, Chap. 2], and [5, Section 8.A].

First of all  $C(q)$  is a  $\mathbb{Z}/2$ -graded algebra; its 0th component is a subalgebra called *even Clifford algebra* and denoted by  $C_0(q)$ . The dimension of  $C(q)$  over  $k$  equals  $2^n$  and the dimension of  $C_0(q)$  is  $2^{n-1}$ , where  $n = \dim_k V$ .

We can consider any vector  $v \in V$  as an element of  $C(q)$ . Clearly, the algebra  $C(q)$  is generated by these elements  $v$  ( $v \in V$ ), while  $C_0(q)$  is the subalgebra generated by the products  $v \cdot v'$  ( $v, v' \in V$ ). Moreover, if  $e_1, \dots, e_n \in V$  is a basis of the vector space  $V$ , then the elements  $e_i$  ( $1 \leq i \leq n$ ) generate  $C(q)$  while the products  $e_i \cdot e_j$  generate  $C_0(q)$ . The generators  $e_i \in C(q)$  are subject to the relations  $e_i^2 = q(e_i)$  ( $1 \leq i \leq n$ ) and  $e_i e_j + e_j e_i = b_q(e_i, e_j)$  ( $1 \leq i, j \leq n, i \neq j$ ), where  $b_q$  is the bilinear form associated to  $q$  (see [4, Definition 1.5 of Chap. I]). If  $\text{char } k \neq 2$ , then there exists an orthogonal basis  $e_1, \dots, e_n$  of  $(V, q)$  and the previous relations become  $e_i^2 = q(e_i)$  and  $e_i e_j = -e_j e_i$ .

There is a unique involution on  $C(q)$  which leaves all the elements  $v$  invariant. It is called the *canonical involution* on the Clifford algebra.

The definition of the *even Clifford algebra* for a quadratic space can be generalized to the case of a central simple algebra  $B$  endowed with a quadratic pair  $\tau$  (see [5, Section 8.B] for the definition). What is obtained in this way is denoted by  $C(B, \tau)$  and called the *Clifford algebra* of  $(B, \tau)$ . Note that  $C(B, \tau)$  is an algebra endowed with an involution. It generalizes the classical notion of even Clifford algebra in the following sense:

**Proposition 2.1** (Knus et al. [5, Proposition 8.8]). *Let  $(V, q)$  be a quadratic space over  $k$  (of even dimension if  $\text{char } k = 2$ ). Let  $B = \text{End}_k(V)$  and let  $\tau$  be the quadratic pair associated to  $q$ . Then there is an isomorphism of algebras with involutions  $C(B, \tau) \simeq C_0(q)$ .*

Denote by  $\mathcal{B}$  the set of isomorphism classes of degree 6 central simple  $k$ -algebras endowed with a quadratic pair. Let  $\mathcal{A}_1$  be the set of isomorphism classes of central simple  $k$ -algebras of degree 4 over a quadratic field extension of  $k$  endowed with a unitary involution trivial on  $k$ . Let  $\mathcal{A}_2$  be the set of isomorphism classes of algebras  $A_2 \times A_2^{op}$  endowed with the exchange involution, where  $A_2$  is a central simple  $k$ -algebra of degree 4 and  $A_2^{op}$  its opposite algebra. We put  $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ . A proof of the following theorem in the characteristic  $\neq 2$  case is given also in [8, Proposition 4].

**Theorem 2.2** (Knus et al. [5, Section 15.D]). *The map  $\mathcal{B} \rightarrow \mathcal{A}$  given by  $(B, \tau) \mapsto C(B, \tau)$  is a bijection. The inverse map  $\mathcal{A} \rightarrow \mathcal{B}$  is given by  $(A, \sigma) \mapsto \mathcal{D}(A, \sigma)$  (see [5, Section 10] for a definition of  $\mathcal{D}$  on  $\mathcal{A}_2$ ).*

### 3. The criterion

Let  $K$  be a field, and let  $A$  be a central simple algebra of degree 4 over  $K$  endowed with an involution  $\sigma$  of unitary type. Let  $k \subset K$  be the subfield of  $\sigma$ -invariant elements of  $K$ . Here is the criterion:

**Theorem 3.1.** *The algebra with involution  $(A, \sigma)$  is decomposable if and only if its discriminant is trivial.*

**Remark 3.2.** Assume that  $\text{char } K \neq 2$ . As we already noticed it in Section 1, the discriminant of  $(A, \sigma)$  coincides with  $D(A, \sigma)$ . Hence,  $(A, \sigma)$  is decomposable if and only if its determinant class modulo 2 is trivial.

**Remark 3.3.** If the exponent of  $A$  is equal to 4, then the algebra itself cannot be decomposed into a tensor product of two proper subalgebras over  $K$ . Hence, we may assume that the exponent of  $A$  divides 2. As mentioned in Section 1, the discriminant of  $(A, \sigma)$  is then represented by an element of  $k^*$ , determined modulo  $N_{K/k}(K^*)$ .

**Remark 3.4.** The criterion is also true for the algebra  $A = A_2 \times A_2^{op}$  with the exchange involution, where  $A_2$  is a central simple  $k$ -algebra of degree 4. In this situation, it is equivalent to a theorem of Albert [1, p. 369] (see also [5, Section 16.1]).

**Proof of Theorem 3.1.** Let  $K, A, \sigma$ , and  $k$  be as introduced in the beginning of this section. We reproduce here a proof of the “only if” part which is already contained in [10, 9] but slightly modified so that it covers the characteristic 2 case. It relies on the following lemma due to Albert:

**Lemma 3.5** (Albert, see [2, Theorem 10.21] or Knus et al. [5, Proposition 2.22]). *Let  $Q$  be a quaternion algebra over  $K$ , endowed with a unitary involution  $\sigma_Q$  trivial on  $k$ . Then there exists a quaternion  $k$ -algebra  $Q_0$  such that  $Q = Q_0 \otimes_k K$  and  $\sigma_Q = \tau \otimes \bar{\phantom{x}}$ , where  $\tau$  is the canonical involution on  $Q_0$  and  $\bar{\phantom{x}}$  the non-trivial automorphism of  $K/k$ .*

Let us assume that the algebra with involution  $(A, \sigma)$  is decomposable. Since  $A$  is of degree 4, it actually decomposes as a tensor product of two quaternion algebras. Hence, because of Albert’s lemma, there exist two quaternion algebras  $Q_1$  and  $Q_2$  over  $k$  such that  $A = (Q_1 \otimes_k Q_2) \otimes_k K$ , and  $\sigma = (\tau_1 \otimes \tau_2) \otimes \bar{\phantom{x}}$ , where  $\tau_i$  is the canonical involution on  $Q_i$ . Therefore, it follows from [5, Proposition 10.33] that the discriminant of  $(A, \sigma)$ , which lies in  $k^*/N_{K/k}(K^*)$  (see Remark 3.3), is given by the discriminant of the orthogonal involution  $\tau_1 \otimes \tau_2$  (see also [9, Proposition 11]). By the decomposability

criterion of Knus–Parimala–Sridharan, the involution  $\tau_1 \otimes \tau_2$  has trivial discriminant and we get  $[\mathcal{D}(A, \sigma)] = 0$ .

Before proving the converse, let us prove the following lemma:

**Lemma 3.6.** *Let  $(V, q)$  be a six-dimensional quadratic  $k$ -space. There is an isomorphism of algebras with involutions  $C_0(q) \simeq Q_1 \otimes_k Q_2 \otimes_k K$  where  $Q_1$  and  $Q_2$  are quaternion  $k$ -algebras,  $K$  is a quadratic étale  $k$ -algebra, and all three are endowed with their canonical involution.*

**Proof.** We break up the proof into two cases:  $\text{char } k = 2$  and  $\text{char } k \neq 2$ . The proof we give in characteristic 2 was suggested by Wadsworth. Let us assume first that  $\text{char } k \neq 2$ . Let  $e_1, \dots, e_6$  be any orthogonal basis of  $V$ , and let  $a_i = q(e_i)$ . As explained in Section 2, the Clifford algebra  $C = C(q)$  is generated by the elements  $e_1, e_2, \dots, e_6$ , which are subject to the relations  $e_i e_j = -e_j e_i$  and  $e_i^2 = a_i$  for all  $i, j$  with  $i \neq j$ . The canonical involution on  $C$ , which we denote by  $\tau$ , is determined by  $\tau(e_i) = e_i$  ( $i = 1, \dots, 6$ ). The even Clifford algebra  $C_0 = C_0(q)$  is the subalgebra of  $C$  generated by the pairwise products of  $e_1, \dots, e_6$ ; the canonical involution  $\tau_0$  on  $C_0$  is the restriction of  $\tau$  to  $C_0$ . In particular, we have  $\tau_0(e_i e_j) = -e_i e_j$  (for  $i \neq j$ ).

Consider the étale  $k$ -algebra  $K = k[t]/(t^2 + a_1 \dots a_6)$  and the quaternion  $k$ -algebras  $Q_1 = (-a_1 a_2, -a_1 a_3)$ ,  $Q_2 = (-a_4 a_5, -a_4 a_6)$ . By definition,  $Q_1$  is generated by elements  $i_1$  and  $j_1$  subject to the relations  $i_1^2 = -a_1 a_2$ ,  $j_1^2 = -a_1 a_3$ , and  $i_1 j_1 = -j_1 i_1$ . The canonical involution on  $Q_1$  is determined by  $i_1 \mapsto -i_1$ ,  $j_1 \mapsto -j_1$ . Analogously,  $Q_2$  is generated by  $i_2$  and  $j_2$  subject to the relations  $i_2^2 = -a_4 a_5$ ,  $j_2^2 = -a_4 a_6$ , and  $i_2 j_2 = -j_2 i_2$ . The canonical involution on  $Q_2$  is determined by  $i_2 \mapsto -i_2$ ,  $j_2 \mapsto -j_2$ . We consider the  $k$ -algebra  $K$  with the involution  $t \mapsto -t$ . One can easily check that the following three homomorphisms of  $k$ -algebras:

$$\begin{array}{lll} Q_1 \rightarrow C_0, & Q_2 \rightarrow C_0, & K \rightarrow C_0 \\ i_1 \mapsto e_1 e_2 & i_2 \mapsto e_4 e_5 & t \mapsto e_1 \dots e_6 \\ j_1 \mapsto e_1 e_3 & j_2 \mapsto e_4 e_6 & \end{array}$$

are well-defined homomorphisms of algebras with involutions and that their images commute. Hence, they induce a homomorphism

$$\psi : Q_1 \otimes_k Q_2 \otimes_k K \rightarrow C_0.$$

If  $K \neq k \times k$ , the algebra  $Q_1 \otimes_k Q_2 \otimes_k K$  is simple. If  $K = k \times k$ , it is not simple anymore, but one can easily check it is indeed simple as an algebra with involution, in the sense it has no non-trivial  $\sigma$ -stable ideal. Since  $\psi$  is a homomorphism of algebras with involution, this proves it is injective in both cases. By a dimension argument, we get that  $\psi$  is an isomorphism.

Assume now that  $\text{char } k = 2$ . The quadratic space  $(V, q)$  can be decomposed into an orthogonal sum  $(V, q) = [a_1, a'_1] \perp [a_2, a'_2] \perp [a_3, a'_3]$ , where  $a_i, a'_i \in k^*$  and  $[a, a']$  denotes the quadratic form  $ax^2 + xy + a'y^2$  (see [4, Proposition 3.4 of Chap. I] or [12, Section 4

of Chap. 9]). Let  $e_1, e'_1, e_2, e'_2, e_3, e'_3$  be the corresponding basis of  $V$ . As explained in Section 2, the Clifford algebra  $C = C(q)$  is generated by the elements  $e_1, e'_1, e_2, e'_2, e_3, e'_3$ , which are subject to the relations  $e_i e_j + e_j e_i = 0$ ,  $e_i e'_j + e'_j e_i = 0$ ,  $e'_i e'_j + e'_j e'_i = 0$ ,  $e_i e'_i + e'_i e_i = 1$ ,  $e_i^2 = a_i$ , and  $e_i'^2 = a'_i$  for all  $i, j$  with  $i \neq j$ . The canonical involution on  $C$ , which we denote by  $\tau$ , is determined by  $\tau(e_i) = e_i$  and  $\tau(e'_i) = e'_i$  ( $i = 1, 2, 3$ ). The even Clifford algebra  $C_0 = C_0(q)$  is the subalgebra of  $C$  generated by the pairwise products of  $e_1, e'_1, e_2, e'_2, e_3, e'_3$ ; the canonical involution  $\tau_0$  on  $C_0$  is the restriction of  $\tau$  to  $C_0$ .

Consider the étale  $k$ -algebra  $K = k[t]/(t^2 + t + a_1 a'_1 + a_2 a'_2 + a_3 a'_3)$  and the quaternion  $k$ -algebras  $Q_1 = [a_1 a'_1, a_1 a_3]$ ,  $Q_2 = [a_2 a'_2, a_2 a_3]$ . By definition (see [13, Section 11 of Chap. 8] or [5, Section 2.C]),  $Q_1$  is generated by elements  $u_1$  and  $v_1$  subject to the relations  $u_1^2 + u_1 = a_1 a'_1$ ,  $v_1^2 = a_1 a_3$ , and  $u_1 v_1 = v_1 u_1 + v_1$ . The canonical involution on  $Q_1$  is determined by  $u_1 \mapsto u_1 + 1$ ,  $v_1 \mapsto v_1$ . Analogously,  $Q_2$  is generated by  $u_2$  and  $v_2$  subject to the relations  $u_2^2 + u_2 = a_2 a'_2$ ,  $v_2^2 = a_2 a_3$ , and  $u_2 v_2 = v_2 u_2 + v_2$ . The canonical involution on  $Q_2$  is determined by  $u_2 \mapsto u_2 + 1$ ,  $v_2 \mapsto v_2$ . We consider the  $k$ -algebra  $K$  with the involution  $t \mapsto t + 1$ . A straightforward computation shows that the homomorphisms of  $k$ -algebras

$$\begin{array}{lll} Q_1 \rightarrow C_0, & Q_2 \rightarrow C_0, & K \rightarrow C_0 \\ u_1 \mapsto e_1 e'_1 & u_2 \mapsto e_2 e'_2 & t \mapsto e_1 e'_1 + e_2 e'_2 + e_3 e'_3 \\ v_1 \mapsto e_1 e_3 & v_2 \mapsto e_2 e_3 & \end{array}$$

are well-defined homomorphisms of algebras with involutions and that their images commute. Hence, they induce a homomorphism

$$\psi : Q_1 \otimes_k Q_2 \otimes_k K \rightarrow C_0.$$

By the same argument as above, this homomorphism  $\psi$  is an isomorphism.  $\square$

Let us now end the proof of Theorem 3.1. Since the degree of  $A$  is 4 (in particular, divisible by 4), the algebra  $\mathcal{D}(A, \sigma)$  is endowed with a quadratic pair  $\tau$ . We assume that the discriminant algebra of  $(A, \sigma)$  is split, i.e. isomorphic to  $\text{End}_k(V)$  for some six-dimensional  $k$ -vector space  $V$ . Thus  $\tau$  is the quadratic pair associated to some quadratic form  $q$  on  $V$  (see [13, Theorem 7.4 of Chap. 8] for the char  $k \neq 2$  case and [5, Proposition 5.11] for the char  $k = 2$  case). By Theorem 2.2, the algebra with involution  $(A, \sigma)$  is isomorphic to the Clifford algebra  $C(\text{End}_k(V), \tau_q)$ ; the latter is isomorphic to  $C_0(q)$  by Proposition 2.1. By Lemma 3.6, the algebra with involution  $C_0(q)$  is decomposable.  $\square$

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