

CANONICAL DIMENSION OF ORTHOGONAL GROUPS

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Abstract. We prove Berhuy–Reichstein’s conjecture on the canonical dimension of orthogonal groups showing that for any integer $n \geq 1$, the canonical dimension of SO_{2n+1} and of SO_{2n+2} is equal to $n(n+1)/2$. More precisely, for a given $(2n+1)$ -dimensional quadratic form ϕ defined over an arbitrary field F of characteristic $\neq 2$, we establish a certain property of the correspondences on the orthogonal grassmannian X of n -dimensional totally isotropic subspaces of ϕ , provided that the degree over F of any finite splitting field of ϕ is divisible by 2^n ; this property allows us to prove that the function field of X has the minimal transcendence degree among all generic splitting fields of ϕ .

1. Results

Let F be an arbitrary field of characteristic $\neq 2$, ϕ a nondegenerate $(2n+1)$ -dimensional quadratic form over F (with $n \geq 1$), X the orthogonal grassmannian of n -dimensional totally isotropic subspaces of ϕ . The variety X is projective, smooth, and geometrically connected; $\dim X = n(n+1)/2$. We write $d(X)$ for the greatest common divisor of the degrees of all closed points on X .

In this paper, a field extension E/F is called a *splitting field* of ϕ if the Witt index (see [10] for the definition of the Witt index of a quadratic form) of the form ϕ_E is maximal (i.e., equal to n). Note that a field extension E/F is a splitting field of ϕ if and only if the set $X(E)$ is nonempty. We write $d(\phi)$ for the greatest common divisor of the degrees of all finite splitting fields of ϕ .

Clearly, $d(\phi) = d(X)$. Moreover, this integer is a power of 2 not exceeding 2^n . The equality $d(\phi) = 2^n$ holds if, for example, the even Clifford algebra $C_0(\phi)$ of the quadratic form ϕ is a division algebra. Of course, this is so for the $(2n+1)$ -dimensional generic quadratic form $\langle t_1, \dots, t_{2n+1} \rangle$ (defined over the field $F(t_1, \dots, t_{2n+1})$ of rational functions in variables t_1, \dots, t_{2n+1}).

A splitting field L/F of ϕ is called *generic* if it is finitely generated and for any splitting field E/F and any nonzero element $a \in L$ there exists an F -place $f: L \rightarrow E$ such that $f(a)$ is neither 0 nor ∞ . The function field $F(X)$ is a generic splitting field

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of ϕ . In fact, it is even *very generic* in the sense of [1] (where it is also explained how the “very generic” property implies the “generic” one): indeed, if E/F is a splitting field, the variety X_E is rational (as any projective homogeneous variety with a rational point is), and therefore $F(X)$ is contained in a purely transcendental extension of E (namely, in $E(X)$).

Following [1], we define the *canonical dimension* $\text{cd}(\phi)$ of ϕ as the minimum of the transcendence degrees of all generic splitting fields of ϕ (the canonical dimension of SO_{2n+1} is then the maximum of $\text{cd}(\phi)$ when ϕ runs over all $(2n+1)$ -dimensional quadratic forms over (finitely generated) extensions of F ; the canonical dimension of SO_{2n+2} coincides with the canonical dimension of SO_{2n+1} , see [1]).

Our main result here reads as follows.

Theorem 1.1. *If $d(\phi) = 2^n$, then $\text{cd}(\phi) = n(n+1)/2$. In particular,*

$$\text{cd}(\text{SO}_{2n+1}) = \text{cd}(\text{SO}_{2n+2}) = n(n+1)/2.$$

The proof is given in Section 2. It immediately follows from Theorem 1.2 (also proved in Section 2), dealing with correspondences on X . A similar situation occurs in the proof of [1, Theorem 11.3] based on [4, Theorem 2.1] dealing with correspondences on a Severi–Brauer variety. An alternative proof of [4, Theorem 2.1], making use of a degree formula, is given in [8, §7.2]. However, for the similar statement [4, Theorem 6.4] concerning correspondences on quadrics (producing a result [5, Theorem 4.3], similar to Theorem 1.1, on the minimum of transcendence degree of generic *isotropy* fields of a quadratic form), there is no proof making use of a degree formula. In the present article as well, we use neither degree formulas nor Steenrod operations.

Theorem 1.2. *If $d(X) = 2^n$, then the multiplicity of any correspondence $\alpha: X \rightsquigarrow X$ is congruent modulo 2 to the multiplicity of the transpose of α . In particular, any rational map $X \rightarrow X$ is necessarily dominant.*

Here, by a correspondence $X \rightsquigarrow X$ we mean an algebraic cycle on $X \times X$ of dimension $\dim X$. The multiplicity $\text{mult}(\alpha)$ of such a correspondence α is defined by the formula $(\text{pr}_1)_*(\alpha) = \text{mult}(\alpha) \cdot [X]$, where $\text{pr}_1: X \times X \rightarrow X$ is the projection onto the first factor, while $(\text{pr}_1)_*$ is the push-forward homomorphism of the group of algebraic cycles, see [3] (we do not use any equivalence relation on algebraic cycles yet). For the transpose α^t of α , we clearly have $\text{mult}(\alpha^t) \cdot [X] = (\text{pr}_2)_*(\alpha)$. The statement on rational maps is obtained by consideration of the correspondence given by the closure in $X \times X$ of the graph of a given rational map $X \rightarrow X$.

Remark 1.3. Assume that $d(X) = 2^n$. Although we have Theorem 1.2, we do not know whether the variety X is *2-incompressible* in the sense of [8, §7]. Note that the only known proof of p -incompressibility of Severi–Brauer varieties of p -primary division algebras (p is an arbitrary prime), given in [8, §7.2], makes use of a degree formula, while the incompressibility of quadrics with first Witt index 1 [5, Corollary 3.4] cannot be proved by a degree formula.

In its turn, Theorem 1.2 follows (very similar to the way [4, Theorem 2.1] follows from [4, Corollary 2.3]) from the following computation of the reduced modulo 2 Chow group $\overline{\text{Ch}}(X)$. Recall that $\overline{\text{Ch}}(X)$ is defined as the image of the restriction homomorphism $\text{Ch}(X) \rightarrow \text{Ch}(\overline{X})$ of the usual modulo 2 Chow groups, where \overline{X} is X over an algebraic closure \overline{F} of F (a general reference for Chow groups is [3]).

Proposition 1.4. *If $d(X) = 2^n$, then $\overline{\text{Ch}}^{>0}(X) = 0$.*

The next section starts with the proof of Proposition 1.4.

2. Proofs

In the proof of Proposition 1.4, we are going to use the description of the integral Chow ring $\text{CH}(\overline{X})$ given in [9] (we borrowed this reference from Totaro’s beautiful paper [11]). The graded ring $\text{CH}^*(\overline{X})$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[e_1, \dots, e_n]$ by the ideal generated by the polynomials

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \dots + (-1)^{i-1}2e_1e_{2i-1} + (-1)^i e_{2i}$$

with $i = 1, \dots, n$ (e_i should be understood as 0 for $i > n$ in this formula), where the degree of e_i is i . The element of $\text{CH}(\overline{X})$ corresponding to the class of e_i is a special Schubert class; we still write e_i for it. For any i , the element $2e_i$ is, up to a sign, the i -th Chern class of the tautological vector bundle on the grassmannian, therefore is rational, that is, lies in the integral reduced Chow group $\overline{\text{CH}}(X) \subset \text{CH}(\overline{X})$.

For any subset I of the set $\{1, 2, \dots, n\}$, let us define an element $e_I \in \text{CH}(\overline{X})$ as the product $\prod_{i \in I} e_i$. Defining $|I|$ as $\sum_{i \in I} i$, we have $\text{codim } e_I = |I|$. The element $e_{\{1,2,\dots,n\}}$ of the maximal codimension $1 + 2 + \dots + n = \dim X$ is equal to the class of a rational point.

A basis of the modulo 2 Chow group $\text{Ch}(\overline{X})$ is given by the classes of the elements e_I , where I runs over all subsets of the set $\{1, 2, \dots, n\}$ (in particular, the dimension of $\text{Ch}(\overline{X})$ (as a vector space over $\mathbb{Z}/2\mathbb{Z}$) is equal to 2^n).

Proof of Proposition 1.4. Assume the contrary: there exists a homogeneous element $\alpha \in \overline{\text{CH}}(X)$ of a positive codimension such that $\alpha \pmod{2}$ is a nonzero element of $\overline{\text{Ch}}(X)$. Decompose α in a sum of some e_I (without repetitions) plus 2β with $\beta \in \text{CH}(\overline{X})$; let us fix a set I such that the element e_I occurs in the decomposition. Let J be the complement of I . Let m be the number of elements in J (note that $m < n$). Then the product $2^m e_J$ is rational. We claim that the degree of the rational 0-cycle $\alpha \cdot (2^m e_J)$ is an odd multiple of 2^m : indeed, the product $e_I \cdot (2^m e_J) = 2^m e_{\{1,2,\dots,n\}}$ has the degree 2^m , while the product $e_{I'} \cdot (2^m e_J)$, for any $I' \neq I$ with $|I'| = |I|$ as well as the product $(2\beta) \cdot (2^m e_J)$, are 0 modulo 2^{m+1} . We now have a contradiction with the assumption on $d(X)$. \square

In the proof of Theorem 1.2, which follows, we use a motivic decomposition of $X \times X$ (in the category of the integral Chow motives), produced in [2]. This motivic decomposition arises from the relative cellular structure on $X \times X$, where the cells are the orbits of the diagonal G -action for $G = \text{SO}(\phi)$. Every summand of this decomposition is a Tate twist of the motive of X . More precisely, there is one copy of the motive of X (without twist, that is, with zero twist), while the remaining summands have some positive twists (although we do not need the precise description of the positive twists, here it is: for any i , the number of summands twisted i times is equal to the rank of the group $\text{CH}_i(\overline{X})$).

To be absolutely precise, we have to say that the motivic decomposition of X given in [2] is not yet the decomposition described above: it also contains motives of certain flag varieties of the tautological vector bundle on X . However, the motive of each such flag variety decomposes in the sum of some twists of the motive of X by [7].

Proof of Theorem 1.2. First, since X is projective, the multiplicity homomorphism factors through the Chow group so that we have $\text{mult} : \text{CH}_N(X \times X) \rightarrow \mathbb{Z}$, where $N = \dim X = n(n+1)/2$. Since the multiplicity of a cycle is not changed under extensions of the base field, the multiplicity homomorphism factors even through the reduced Chow group, so that we may replace $\text{CH}(X \times X)$ by $\overline{\text{CH}}(X \times X)$. Since we are interested in multiplicities modulo 2, we consider the induced homomorphism of the modulo 2 Chow group (still denoted by mult): $\text{mult} : \overline{\text{Ch}}_N(X \times X) \rightarrow \mathbb{Z}/2\mathbb{Z}$.

The theorem we are trying to prove claims that the image of the homomorphism

$$f : \overline{\text{Ch}}(X \times X) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \quad f : \alpha \mapsto (\text{mult}(\alpha), \text{mult}(\alpha^t))$$

is contained in the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Using the above description of motivic decomposition of $X \times X$, we get a decomposition of $\overline{\text{Ch}}_N(X \times X)$ in the direct sum, where the summands are: one copy of $\overline{\text{Ch}}_N(X)$ and several copies of $\overline{\text{Ch}}_i(X)$ with various $i < N$. Since $\overline{\text{Ch}}_i(X) = 0$ for any $i < N$ by Proposition 1.4, the image of the homomorphism f is cyclic. Since, on the other hand, $f([\Delta_X]) = (1, 1)$, the image of f is generated by $(1, 1)$, that is, coincides with the diagonal subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. \square

Proof of Theorem 1.1. We repeat the proof of [1, Theorem 11.3] using Theorem 1.2 instead of [8, §7.2] (and meaning by X our orthogonal grassmannian instead of a Severi–Brauer variety).

Since the field $F(X)$ is a generic splitting field of ϕ and has the transcendence degree $n(n+1)/2$, the inequality $\text{cd}(\phi) \leq n(n+1)/2$ holds (the assumption on $d(\phi)$ is not needed for this bound).

If L is now another generic splitting field of ϕ , then we show that $\text{tr. deg}(L/F) \geq n(n+1)/2$ as follows. Let Y be a projective model of L/F . Since both $F(X)$ and $F(Y)$ are generic splitting fields of ϕ , there exist rational morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$. Moreover, for any nonempty open subset $U \subset Y$, there exists a rational morphism $X \rightarrow Y$ with an image meeting U , so that we may assume that f and g are composable. Since the rational map $X \rightarrow X$ given by the composition $g \circ f$ is dominant by Theorem 1.2, the dimension of Y is at least equal to $\dim X = n(n+1)/2$. \square

Remark 2.1. Let us say that a splitting field L/F of ϕ is *quasigeneric* if it simply has a place to any splitting field of ϕ (see [6]). Then we can define a quasicanonical dimension $\text{qcd}(\phi)$ as the minimal transcendence degree of a quasigeneric splitting field. Clearly, $\text{qcd}(\phi) \leq \text{cd}(\phi)$. But in the situation of Theorem 1.1, we can show that the equality holds. Namely (see [5, proof of Theorem 4.3]), if Y is a projective model of a quasigeneric splitting field of ϕ , the places $F(X) \rightarrow F(Y)$ and $F(Y) \rightarrow F(X)$ produce rational morphisms $Y \rightarrow X$ and $X \rightarrow Y$; replacing Y by the closure in $Y \times X$ of the graph of the rational morphism $Y \rightarrow X$, we get a model of the same field possessing a regular morphism to X , so that we can compose it with the rational morphism $X \rightarrow Y$ and conclude that $\dim Y \geq \dim X$ by Theorem 1.2. Similarly, $\text{qcd}(A) = \text{cd}(A)$ for any p -primary central division algebra A .

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