

Clifford Modules and Twisted K -Theory

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Abstract. The purpose of this short paper is to make the link between the fundamental work of Atiyah, Bott and Shapiro [ABS] and twisted K -theory [DK] [R] [K5]. This link was implicit for a long time in the literature (for the description of the real K -theory of spheres as an example) but was not explicitly defined before.

Keywords. Clifford modules, twisted K -theory, Bott periodicity, Thom spaces.

The setting is the following: V is a real vector bundle on a compact space X , provided with a non degenerate quadratic form to which we associate a bundle of (real or complex) Clifford algebras denoted by $C(V)$; the quadratic form is implicit in this notation. We denote by $M(V)$ the Grothendieck group associated to the category of (real or complex) vector bundles provided with a structure of (twisted) $\mathbb{Z}/2$ -graded $C(V)$ -module. Another way to describe $M(V)$ is to consider the bundle $V \oplus 1$, where the symbol “1” denotes the trivial vector bundle of rank one with a positive quadratic form. Then $M(V)$ is just the Grothendieck group $K(\Lambda_1)$ of the category $\mathcal{P}(\Lambda_1)$ of finitely generated projective modules over Λ_1 . Here, the notation Λ_n means in general $\Lambda \widehat{\otimes} C^{0,n}$, where Λ is the ring of continuous sections of the $\mathbb{Z}/2$ -graded bundle $C(V)$ and $C^{0,n}$ is the Clifford algebra of \mathbb{R}^n with a positive quadratic form.

Following [ABS], we now define $A(V)$ as the cokernel of the homomorphism induced by restriction of the scalars :

$$A(V) = \text{Coker}[M(V \oplus 1) \rightarrow M(V)] = \text{Coker}[K(\Lambda_2) \rightarrow K(\Lambda_1)].$$

Remark

Let us denote by V^- the vector bundle V with the opposite quadratic form. It is quite easy to see¹ that the category of $C(V \oplus 1)$ -modules is isomorphic to the category of $C(V^- \oplus 1)$ -modules. From now on, we assume that the quadratic form on V is positive (in which case $A(V^-)$ was the original definition of [ABS]).

¹If (v, t) is a symbol for the action of $V \oplus 1$, with $t^2 = 1$, we change it in (vt, t) which represents the action of $V^- \oplus 1$.

From these definitions, we get the following theorem, where $K(V)$ denotes the real or complex reduced K -theory of the Thom space of V .

Theorem

There is an exact sequence between K -groups²

$$K(\Lambda_2) \rightarrow K(\Lambda_1) \rightarrow K(V) \rightarrow K^1(\Lambda_2) \rightarrow K^1(\Lambda_1)$$

In particular, $A(V)$ is a subgroup of $K(V)$ which coincides with it in the following important cases :

- a) $K^1(\Lambda_2) = 0$, for instance when X is reduced to a point.
- b) V is oriented of rank divisible by 4.
- c) V is oriented of even rank in the framework of complex K -theory.

Proof. According to the general theory developed in [K1], $K(V) \cong K^1(V \oplus 1)$ is canonically isomorphic to the K^1 -group of the Banach functor

$$\phi : \mathcal{E}^{V \oplus 2}(X) \rightarrow \mathcal{E}^{V \oplus 1}(X),$$

where $\mathcal{E}^W(X)$ denotes the category of vector bundles provided with a $C(W)$ -module structure. According to the Serre-Swan theorem, the categories involved are equivalent to categories $\mathcal{P}(R)$ for suitable rings R , in this case Λ_2 or Λ_1 . The first part of the theorem follows from these general considerations.

If X is a point, the category $\mathcal{P}(R)$ is finite dimensional and therefore its K^1 -group is trivial. On the other hand, if V is oriented of rank n divisible by 4, let us choose an orthonormal oriented basis e_1, \dots, e_n on each fiber $V_x, x \in X$. Then the product $\varepsilon = e_1 \dots e_n$ in the Clifford algebra $C(V_x)$ is independent of the choice of the basis since ε commutes with the action of $SO(n)$ and defines therefore a continuous section of $C(V)$. On the other hand, for any W , there is an isomorphism between the graded tensor product $C(V) \hat{\otimes} C(W)$ and the nongraded one $C(V) \otimes C(W)$. For this purpose, we send first $V \oplus W$ to $C(V) \otimes C(W)$ by the formula

$$(v, w) \mapsto v \otimes 1 + \varepsilon \otimes w,$$

The fact that n is even shows that ε anticommutes with v . Moreover, if 4 divides n , the square of ε is equal to 1. Therefore, by the universal property of Clifford algebras, the previous map induces the required isomorphism $C(V) \hat{\otimes} C(W) \cong C(V) \otimes C(W)$. If $n = 4k + 2$ and in the framework of complex K -theory, one may replace ε by $\varepsilon\sqrt{-1}$ in order to get the same result.

In our situation, W is of dimension one or two and the Banach functor

$$\mathcal{P}(\Lambda) \sim \mathcal{P}(\Lambda \hat{\otimes} C^{0,2}) \rightarrow \mathcal{P}(\Lambda \hat{\otimes} C^{0,1}) \sim \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda)$$

may be identified to the diagonal functor through the previous category isomorphisms. This shows that the map

$$K^1(\Lambda \hat{\otimes} C^{0,2}) \rightarrow K^1(\Lambda \hat{\otimes} C^{0,1})$$

is injective and concludes the proof of the theorem. □

²One might also write K_{-1} instead of K^1 .

Example

When X is reduced to a point, the theorem implies that the reduced K -theory of the sphere S^n is the cokernel of the map

$$K(C^{0,n+2}) \rightarrow K(C^{0,n+1})$$

which is the same as the cokernel of the map

$$K(C^{n+1,1}) \rightarrow K(C^{n,1})$$

as we noticed earlier. This is the starting remark in [ABS] which was the inspiration of [K1], where the notation $C^{p,q}$ is used.

Generalizations

Since the main tool used here is the real Thom isomorphism proved in [K1] and [K3], the previous theorem might be generalized to the equivariant case. For instance, if G is a finite group acting linearly on \mathbb{R}^n , the group $K_G(\mathbb{R}^n)$ is isomorphic to the cokernel of the following map

$$K(G \ltimes C^{0,n+2}) \rightarrow K(G \ltimes C^{0,n+1})$$

where the involved rings are crossed products of G by Clifford algebras. More precise results may be found in [K4].

Another generalization is to consider modules over bundles of $\mathbb{Z}/2$ -graded Azumaya algebras \mathcal{A} as in [DK], instead of bundles of Clifford algebras. The analog of the group $A(V)$ is now what we might call the “algebraic twisted K -theory” of \mathcal{A} , denoted by $K_{alg}^{\mathcal{A}}(X)$ which is the cokernel of the map $K(\Lambda_2) \rightarrow K(\Lambda_1)$, where Λ denotes the ring of continuous sections of the $\mathbb{Z}/2$ -graded algebra bundle \mathcal{A} . We can prove, as in the previous theorem, that this new group is a subgroup³ of the usual twisted K -theory of X denoted by $K^{\mathcal{A}}(X)$. It coincides with it in some important cases, for instance if \mathcal{A} is oriented (in the graded sense) with fibers modelled on matrix algebras over the real or complex numbers.

The multiplicative structure

It is well known that the twisted K -groups $K^{\mathcal{A}}(X)$ can be provided with a cup-product structure (see [DK] §7 and also [K2]). This cup-product makes a heavy use of Fredholm operators in Hilbert spaces. These Hilbert spaces are unavoidable, especially for the odd K -groups.

More precisely, as shown in [DK], the elements which define the group $K^{\mathcal{A}}(X)$ are pairs (E, D) where E is a $\mathbb{Z}/2$ -graded Hilbert bundle provided by an \mathcal{A} -module structure and $D : E \rightarrow E$ is a family of Fredholm operators which are self-adjoint, of degree one and commute (in the graded sense) with the action of \mathcal{A} . According to the Thom isomorphism in twisted K -theory [K5], $K^{\mathcal{A}}(X)$ is also the K^1 -group of the Banach functor

$$\phi : \mathcal{E}^{\mathcal{A}} \hat{\otimes} C^{0,2}(X) \rightarrow \mathcal{E}^{\mathcal{A}} \hat{\otimes} C^{0,1}(X).$$

³One has to use again the Thom isomorphism in twisted K -theory as stated in [K5].

We have therefore the following exact sequence (as for Clifford modules)

$$K(\mathcal{A} \widehat{\otimes} C^{0,2}) \rightarrow K(\mathcal{A} \widehat{\otimes} C^{0,1}) \rightarrow K^{\mathcal{A}}(X) \rightarrow K^1(\mathcal{A} \widehat{\otimes} C^{0,2}) \rightarrow K^1(\mathcal{A} \widehat{\otimes} C^{0,1}).$$

A closer look at the connecting homomorphism

$$K(\mathcal{A} \widehat{\otimes} C^{0,1}) \rightarrow K^{\mathcal{A}}(X)$$

shows that it associates the pair $(E, 0)$ to a (finite dimensional) vector bundle E which is a module over $A \widehat{\otimes} C^{0,1}$. In other words, the elements in $K^{\mathcal{A}}(X)$ associated to finite dimensional bundles E are just elements of the cokernel of the map $K(\mathcal{A} \widehat{\otimes} C^{0,2}) \rightarrow K(\mathcal{A} \widehat{\otimes} C^{0,1})$ which we might call $A(\mathcal{A})$, if we follow the conventions of [ABS] or simply $K_{alg}^{\mathcal{A}}(X)$ as we did before.

Since the elements of $K_{alg}^{\mathcal{A}}(X)$ are associated to finite dimensional bundles (with the Fredholm operator reduced to 0), the usual cup-product

$$K^{\mathcal{A}}(X) \times K^{\mathcal{A}'}(X) \rightarrow K^{\mathcal{A} \widehat{\otimes} \mathcal{A}'}(X)$$

induces a pairing between the algebraic parts

$$K_{alg}^{\mathcal{A}}(X) \times K_{alg}^{\mathcal{A}'}(X) \rightarrow K_{alg}^{\mathcal{A} \widehat{\otimes} \mathcal{A}'}(X).$$

On the other hand, it might be interesting to characterize the elements of $K^{\mathcal{A}}(X)$ which are “algebraic”. They belong to the kernel of the map

$$\phi : K^{\mathcal{A}}(X) \rightarrow K^1(\mathcal{A} \widehat{\otimes} C^{0,2}) = K(\mathcal{B}/\mathcal{A} \widehat{\otimes} C^{0,2}).$$

The notation \mathcal{B}/\mathcal{A} represents here the bundle of Calkin algebras associated to \mathcal{A} (the structural group of \mathcal{A} is $PU(H)$ where H is an infinite dimensional Hilbert space, as mentioned in [K5] and [R]). This map ϕ is easy to describe: it associates to a couple (E, D) as before the space of sections of the \mathcal{B}/\mathcal{A} -bundle associated to E . It is provided with the action of $C^{0,2}$ described by the grading and the involution on the Calkin bundle induced by the polar decomposition of D . This description of the “algebraic” elements also holds in the generalization of twisted K -theory considered by J. Rosenberg [R][K5].

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