THETA CORRESPONDENCES FOR CLOSE UNITARY GROUPS

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to Steve Kudla: collaborator, teacher, and friend

Introduction

This is a sequel to the articles [HKS], in which the local theory of the theta correspondence is developed for unitary groups over p-adic fields; [H3, H5], in which this local theory is made more precise and applied to obtain results about special values of L-functions and period relations; and [L1, L2], which work out the local archimedean theory for cohomological representations and the connection to special values of L-functions in the stable range. The goal of the present paper is to refine the earlier results and to outline the general global theory of the theta correspondence for cuspidal automorphic representations of unitary groups of discrete series type at infinity, when the groups are roughly the same size. We consider a quadratic extension F/F^+ of number fields, with F^+ totally real and F totally imaginary, and a dual reductive pair of groups over F^+ of the form $(G, G') = (U(W), U(V))$, where W and V are hermitian vectors spaces over F, $\dim W = n$, $\dim V = m$.

The non-triviality of the local theta correspondence for these pairs is considered in [HKS] and [H5], as well as [GG]. The non-triviality of the global theta correspondence is determined by the local correspondence and by a special value of an L-function, which is at the central point if $m = n$ and at the near central point if $m = n + 1$; cf. [H5, 3.3, 3.4]. The connection is provided by the extended Siegel-Weil formula, due to Ichino, and the corresponding Rallis inner product formula. Bearing in mind the expressions for special values of these L-functions in terms of periods of arithmetic automorphic forms, we formulate a conjecture on the rationality of the theta correspondence when $m = n$. This conjecture extends the results proved in [H3] to general pairs of

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discrete series type, and has the peculiarity of being quite concrete but apparently completely inaccessible by all available techniques. We will have more to say about this below.

The present article concentrates on the theta correspondence when $m = n$, which parallels the theory of stable transfer of automorphic representations between inner forms of the same group. This theory has been largely worked out under simplifying hypotheses, mainly by Labesse [Lab]. The property of local dichotomy, conjectured in [HKS] and proved in [H5] and [GG], provides an explicit formula for the central value in terms of inner products and local zeta integrals. In some cases we can use this formula to show that the central value is non-negative, as predicted by the generalized Riemann hypothesis. See [KZ, KS, G, LR1] for other approaches to this question. The argument given here seems particularly simple.

It is indeed a great pleasure to dedicate this article to Steve Kudla. Nearly thirty years ago, one of us began a collaboration with Steve that included many of the highlights of his career and extends well beyond the papers published under our two names. The second-named author feels extremely lucky to have Steve as a wonderful friend and mentor for over twenty years. All three authors have benefited immeasurably from his imagination, clarity, and patience.

Preliminary notation

Let F^+ be a totally real field, F a totally imaginary quadratic extension of F^+. We will often assume F contains a chosen imaginary quadratic field, which is then denoted $\mathcal{K}$, and then $F = \mathcal{K} \cdot F^+$. The quadratic Hecke character of $\mathbf{A}_{F^+}^\times$ corresponding to the extension F/F^+ is denoted

$$\eta_{F/F^+} : \mathbf{A}_{F^+}^\times / F^{+,\times} N_{F/F^+} \mathbf{A}_F^\times \xrightarrow{\sim} \pm 1.$$

Let W be an n-dimensional F-vector space, endowed with a non-degenerate hermitian form $< \bullet, \bullet >_W$, relative to the extension F/F^+. We let Σ^+, resp. Σ_F, denote the set of complex embeddings of F^+, resp. F, and choose a CM type $\Sigma \subset \Sigma_F$, i.e. a subset which upon restriction to F^+ is identified with Σ^+. Complex conjugation in $\mathrm{Gal(F/F^+)}$ is denoted c.

The hermitian pairing $< \bullet, \bullet >_W$ defines an involution $\tilde{c}$ on the algebra $End(W)$ via

$$< a(v), v' >_W = < v, a^{\tilde{c}}(v') >_W, \tag{0.1}$$

and this involution extends to $End(W \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define the algebraic group $U(W) = U(W, < \bullet, \bullet >_W)$ over F^+ such that, for any F^+-algebra R,

$$U(W)(R) = \{g \in GL(V \otimes_{F^+} R) \mid g \cdot \tilde{c}(g) = 1\}; \tag{0.2}$$

All constructions relative to hermitian vector spaces carry over without change to skew-hermitian spaces. If $(W, < \bullet, \bullet >_W)$ is a hermitian space, let $Ł \in F$ be an element with $Tr_{F/F^+}(Ł) = 0$. Then $Ł \cdot < \bullet, \bullet >_W$ is a skew-hermitian form whose symmetry group is exactly $U(W)$.

A general quadratic extension of local fields is denoted L/K, except when the extension is obtained by specializing a quadratic extension of number fields. The quadratic character is denoted $\eta_{L/K}$. For an n-dimensional hermitian space W/L we define $U(W)$ as above. The case $L = K \oplus K$ is admitted; then $U(W) \xrightarrow{\sim} GL(n)_K$.

In §3 and §4 we use the notation $\mathbf{n}$ to denote the set $\{1, \ldots, n\}$.

For the construction of the oscillator representation, we fix a nontrivial additive character $\psi : \mathbf{A}_{F^+} \to \mathbb{C}^\times$.

1. Conventions for the theta correspondence

1.1. **The local correspondence.** Let L/K be a quadratic extension of local fields, and W and V hermitian spaces over L as above. We fix an additive character $\psi : K \to \mathbb{C}^\times$. We briefly recall the outlines of the theory [KS, HKS] of the local theta correspondence from irreducible admissible representations of $G = U(W)$ to representations of $U(V)$, where $(V, (,)_V)$ is a variable hermitian space over K of dimension m.

As in [KS] and [HKS] we choose (a pair of) splitting character(s) $\chi = (\chi_1, \chi_2)$ of $K^\times$ satisfying

$$\chi_1 \mid_{K^\times} = \eta_{L/K}^m, \quad \chi_2 \mid_{K^\times} = \eta_{L/K}^n. \tag{1.1.1}$$

When $m = n$ we will always take $\chi_1 = \chi_2$, and we abuse notations slightly to write $\chi = (\chi, \chi)$. The choice of splitting character χ, together with the additive character ψ, defines a Weil representation $\omega_{V,W,\chi}$ of the dual reductive pair $U(W) \times U(V)$ on an appropriate Schwartz-Bruhat space $\mathcal{S}_{V,W}$. For details, see [HKS] for K p-adic, and [Pa] for $K = \mathbb{R}$.

Let π be an irreducible admissible representation of G. We define the representation $\Theta_\chi(V, \pi)$ of $U(V)$ by

$$\Theta_\chi(V, \pi) = [\omega_{V,W,\chi} \otimes \pi]_G = Hom_G(\omega_{V,W,\chi}^\vee, \pi). \tag{1.1.2}$$

Note that this is the full theta correspondence, which often gives a result larger than the Howe correspondence.

The Howe quotient of $\Theta_\chi(V, \pi)$ is the sum of all its $U(V)$-irreducible quotients. *Howe duality* is the assertion that the Howe quotient of $\Theta_\chi(V, \pi)$ is irreducible or zero for all V, π, χ. This has been proved for all local fields (including archimedean local fields) of residue characteristic different from 2, for $GL(n)$ in general, and for *generic* π in an appropriate sense even if the residue characteristic is 2. We will assume it holds in all the cases we consider.

1.2. **The global correspondence.** In this section we choose an element Ł $\in F$ as in the notation section to convert W to a skew-hermitian space. The space $\mathbb{W} = R_{F/F^+}V \otimes W$ then naturally carries a symplectic form $< \bullet, \bullet >_{\mathbb{W}}$. Let $\widetilde{Sp}(\mathbb{W})$ be the (metaplectic) covering of $Sp(\mathbb{W})$ with kernel $\mathbb{C}^\times$:

$$1 \to \mathbb{C}^\times \to \widetilde{Sp}(\mathbb{W})(\mathbf{A}_{F^+}) \to Sp(\mathbb{W})(\mathbf{A}_{F^+}) \to 1 \tag{1.2.1}$$

Together with the fixed additive character ψ, a choice of complete polarization

$$\mathbb{W} = \mathbb{X} \oplus \mathbb{Y} \tag{1.2.2}$$

then defines a Schrödinger model of the oscillator representation of $\widetilde{Sp}(\mathbb{W})(\mathbf{A}_F)$

$$\omega_\psi : \widetilde{Sp}(\mathbb{W})(\mathbf{A}_{F^+}) \to Aut(\mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))), \tag{1.2.3}$$

where $\mathcal{S}$ denotes the space of Schwartz-Bruhat functions.

Let $G = U(W)$, $G' = U(V)$. We now choose a global splitting character χ of $\mathbf{A}_F^\times/F^\times$ satisfying the global analogue of (1.1.1). As above, ω_ψ then defines a Weil representation $\omega_{V,W,\chi}$ of $U(W)(\mathbf{A}) \times U(V)(\mathbf{A})$ on the Schwartz-Bruhat space $\mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))$. For any $\Phi \in \mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))$, we then define the theta kernel as usual

$$\theta_\Phi(g,h) = \sum_{\lambda \in \mathbb{X}(F^+)} \omega_{V,W,\chi}(g,h)(\Phi)(\lambda), \tag{1.2.4}$$

with $g \in G(\mathbf{A}_{F^+}), h \in G'(\mathbf{A}_{F^+})$. Then for any cusp form

$$f : G(F^+)\backslash G(\mathbf{A}_{F^+}) \to \mathbb{C},$$

we define

$$\theta_\Phi(f)(g') = \int_{G(F^+)\backslash G(\mathbf{A}_{F^+})} \theta_\Phi(g,g')f(g)dg, \quad g' \in G'(\mathbf{A}_{F^+}) \tag{1.2.5}$$

where dg is Tamagawa measure.

Let π be a cuspidal automorphic representation of G. The automorphic representation $\Theta_\chi(V,\pi)$ of G' is the space of all $\theta_\Phi(f)$ as f and Φ vary over π and $\mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))$, respectively.

1.3. L-functions and the Rallis inner product formula. Note that formula (1.2.5) is $\mathbb{C}$-linear in f and ϕ. In this section we recall the Rallis inner product formula in the case $m = n$, where it is established in [H5, §3.5], using the results of [I1] on the Siegel-Weil formula. The statement of Theorem 3.5.2 of [H5] includes the hypothesis that the space V is positive definite, in which case the Siegel-Weil formula is in no need of regularization. This hypothesis, which sufficed for the purposes of [H5], is in fact unnecessary, because the regularized Siegel-Weil formula has been established as Theorem 4.2 of [I1].

To state the Rallis inner product formula, we need to recall the construction of the standard L-function for unitary groups. We refer to [H5] for terms introduced here without explanation. Let $H = U(W \oplus (-W))$ be the quasi-split unitary group of size $2n$, as in the doubling method (cf. [HKS] for example), and let $i_W : G \times G = U(W) \times U(-W) \hookrightarrow H$ be the natural inclusion. Let $\pi, \pi^\flat$ be cuspidal automorphic representations of $G = U(W) = U(-W)$, $f \in \pi$, $f^\flat \in \pi^\flat$. Define the Piatetski-Shapiro-Rallis zeta integral

$$Z(s, f, f^\flat, \varphi) = \int_{(G\times G)(F^+)\backslash(G\times G)(\mathbf{A})} E(i_W(g, g^\flat), s, \varphi, \chi) f(g) f^\flat(g^\flat) \chi^{-1} \circ \det(g^\flat) dg dg^\flat$$

with $E(\bullet, s, \varphi) = E(\bullet, s, \varphi, \chi)$ the Eisenstein series on $H(\mathbf{A}_{F^+})$ as in [H5, §1]; here χ satisfies (1.1.1), with $m = n$, and $\varphi = \varphi(s)$ is an element of a degenerate principal series representation $I_n(s, \chi)$ varying with s. This integral is absolutely convergent for $Re(s)$ sufficiently large and admits an Euler product expansion if the section φ and the vectors f and $f^\flat$ are factorizable. The integral vanishes unless $\pi^\flat \xrightarrow{\sim} \pi^\vee$, which we will assume in what follows.

If $f, f^\flat$ are cusp forms on $G(\mathbf{A})$, we write

$$< f, f^\flat >= \int_{G(F^+)\backslash G(\mathbf{A})} f(g) f^\flat(g) dg,$$

so that $(f, f^\flat) \mapsto < f, \bar{f^\flat} >$ is the L^2 pairing. Write $\pi = \otimes \pi_v$ and $\pi^\vee = \otimes \pi_v^\vee$. For each place v we fix a local bilinear invariant pairing $< \bullet, \bullet >_v$ between π_v and $\pi_v^\vee$. We make compatible choices so that if $f = \otimes f_v$ and $f^\flat = \otimes f_v^\flat$ then

$$< f, f^\flat >= \prod < f_v, f_v^\flat >_v$$

Since each π_v is unitary, there is a conjugate linear isomorphism

$$\pi_v \longrightarrow \pi_v^\vee, \qquad f_v \mapsto \bar{f}_v$$

We make compatible choices so that $(f_v, g_v) = < f_v, \bar{g}_v >_v$ is the local invariant inner product, and that $\bar{f} = \otimes \bar{f}_v$ if $f = \otimes f_v$.

For S a sufficiently large set of primes of F^+, including the archimedean primes, and writing $f = \otimes_v f_v$, $f^\flat = \otimes_v f_v^\flat$ and $\varphi = \otimes_v \varphi_v$, the Euler product looks like this:

(1.3.1)
$$Z(s, f, f^\flat, \varphi, \chi) = \prod_{v \in S} Z_v(s, f_v, f_v^\flat, \varphi_v, \chi_v) \cdot d_n^S(s)^{-1} L^S(s + \frac{1}{2}, \pi, St, \chi).$$

Here $L^S(s + \frac{1}{2}, \pi, St, \chi)$ is the product over primes not in S of the standard (unramified) Euler factors for the group G, twisted by χ,

$$d_n(s) = \prod_{r=0}^{n-1} L(2s + n - r, \eta_{F/F^+}^{n+r}) \tag{1.3.2}$$

attached to the $2n$-dimensional representation of the L-group (cf. [H5] for notation), and d_n^S is d_n with factors at S removed. We write

(1.3.3)
$$\tilde{Z}_v(s, f_v, f_v^\flat, \varphi_v, \chi_v) = Z_v(s, f_v, f_v^\flat, \varphi_v, \chi_v) d_{n,v}(s) L_v(s + \frac{1}{2}, \pi_v, St, \chi_v)^{-1}$$

with the local L-factor defined as in [HKS], so that (1.3.1) can be rewritten more neatly

(1.3.4)
$$Z(s, f, f^\flat, \varphi, \chi) = \prod_{v \in S} \tilde{Z}_v(s, f_v, f_v^\flat, \varphi_v, \chi) \cdot d_n(s)^{-1} L(s + \frac{1}{2}, \pi, St, \chi)$$

Now we choose $\phi = \otimes \phi_v \in \mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))$ as in (1.2). Since π is a cuspidal representation of G, it is unitary; hence

$$\pi^\flat \xrightarrow{\sim} \pi^\vee \Rightarrow \pi^\flat \xrightarrow{\sim} \bar{\pi}$$

(complex conjugate). It is thus legimitate to take $f^\flat = \bar{f}$, and we do so henceforward. The *Rallis inner product formula* is then

(1.3.5)
$$\frac{< \theta_\phi(f), \theta_{\bar{\phi}}(\bar{f}) >}{< f, \bar{f} >} = \prod_{v \in S} \frac{\tilde{Z}_v(0, f_v, f_v^\flat, \varphi_v, \chi)}{< f_v, \bar{f}_v >_v} \cdot d_n(0)^{-1} L(\frac{1}{2}, \pi, St, \chi).$$

Here φ_v is an element of the degenerate principal series constructed explicitly in terms of the pair $(\phi_v, \bar{\phi}_v)$ of local Schwartz-Bruhat functions, as we explain in §4. The values at 0 on the right-hand side are defined by analytic continuation. The left-hand side of (1.3.5) is clearly a positive real number if it does not vanish. We return to this point in (4.2).

2. The archimedean theta correspondence

2.1. **Review of the results for close unitary groups.** We consider two unitary groups $G = U(n)$ and $G' = U(m)$ over $\mathbb{R}$ with signatures to be determined later. We will assume $m \geq n$. The theta lifts of discrete series representations of $U(n)$ to cohomological representations of $U(m)$ were determined explicitly in [L2] in great generality. The definition of the theta correspondence for real unitary groups depends on the choice of a character

$$\psi_a : \mathbb{R} \to U(1),\ t \mapsto e^{2a\pi it}$$

for some $a \in \mathbb{R}$. The characters ψ_a and $\psi_{a'}$ define the same correspondence of representations if and only if $a/a' > 0$. As in [H5] we take $a > 0$. For the purposes of this section we may assume $a = 1$, though it makes no difference. For global applications, where there are several real embeddings σ of the field F^+ we need to be free to let $a = a_\sigma$ be the image under σ of a totally positive element of F^+.

We identify the unitary groups G, G' with the classical unitary groups of given signatures $G = U(s,r)$, $G' = U(p,q)$, with maximal compact subgroups $K = U(s) \times U(r)$, $K' = U(p) \times U(q)$, respectively; thus $r + s = n$, $p + q = m$. The results of [L2] are stated in terms of the the metaplectic representation and determine liftings from a double cover $\tilde{G}$ of G to a double cover $\tilde{G}'$ of G', defined as the pullback of the metaplectic cover of $Sp(2nm)$ under the map $G \times G' \to Sp(2nm)$ defining (G, G') as a dual reductive pair. The corresponding double covers of K and K' are denoted $\tilde{K}$ and $\tilde{K}'$, respectively. These are reinterpreted in terms of the unitary groups using splitting characters, as in [H5]. When m is even, the cover $\tilde{G}$ of G splits, whereas it is obtained from the non-trivial two-fold cover of the center $U(1)$ of G if m is odd. This property is symmetric when the roles of m and n are exchanged.

The infinitesimal characters of finite-dimensional representations of G are indexed as usual by decreasing n-tuples $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ where the λ_i are half-integers if n is even and integers if n is odd. It follows from the discussion above that the infinitesimal characters of finite-dimensional representations of $\tilde{G}$ can be naturally indexed by decreasing n-tuples $\tilde{\lambda}_1 > \tilde{\lambda}_2 \ldots \tilde{\lambda}_n$, where for all i,

$$\tilde{\lambda}_i \equiv \frac{n-1+m}{2} \pmod{\mathbb{Z}}. \tag{2.1.1}$$

In particular, if m is even then we can identify $\tilde{\lambda}$ with λ.

For the theta correspondence, it is best to work with *strong inner forms* in the sense of Vogan, which means that $U(r,s)$ and $U(s,r)$ are considered separately if $r \neq s$. The infinitesimal characters of irreducible finite-dimensional representations of G can be identified with the Langlands parameters ϕ of discrete series of the (strong) inner forms of G by a standard procedure, cf. [Cl] for the formulas. Thus the discrete series L-packet Π_ϕ of G is parametrized by the infinitesimal character $\lambda = \lambda_\rho$ for some finite-dimensional representation ρ, written as a decreasing n-tuple as above, and we write $\Pi_{\lambda,G}$ instead of Π_ϕ. A discrete series L-packet of $\tilde{G}$ is similarly indexed by an infinitesimal character, denoted $\tilde{\lambda}$, whose parameters satisfy (2.1.1). Note that the same λ or $\tilde{\lambda}$ serves for all inner forms of $U(n)$. The disjoint union of the L-packets $\coprod_{U(r,s)} \Pi_{\lambda,U(r,s)}$ is denoted Π_λ. The elements of Π_λ can be identified with their *Harish-Chandra parameters*. These can be written as pairs $(U(r,s),(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_s))$ (the U is superfluous) where

(2.1.2) $\alpha_i > \alpha_{i+1}$, $\beta_j > \beta_{j+1}$ for all i,j;
(2.1.3) The n-tuple $(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s)$ is a permutation of $(\lambda_1,\dots,\lambda_n)$.

In other words, the Harish-Chandra parameters are obtained by taking permutations of the coordinates of λ that can be partitioned by a semicolon into exactly two decreasing sequences. For each $U(r,s)$, the partition $(\lambda_1,\dots,\lambda_r;\lambda_{r+1},\dots,\lambda_{r+s})$ serves as a natural basepoint $\lambda(r,s)_0$. The set $\Pi_{\lambda,U(r,s)}$ is then in bijection with s-tuples $(\beta_1,\dots,\beta_s)$ chosen from the λ_i, or alternatively with (r,s)-shuffles of the parameters of λ; thus $|\Pi_{\lambda,U(r,s)}| = \frac{n!}{r!s!}$, and $|\Pi_\lambda| = 2^n$. We define an explicit identification of Π_λ with $\{\pm 1\}^n$ in §2.2.

The above description remains valid, with the obvious modifications, for the parameters $\tilde{\lambda}$. The theta correspondence is calculated in terms of the corresponding Harish-Chandra parameters. Let $\tilde{\pi} \in \Pi_{\tilde{\lambda},\tilde{U}(r,s)}$, with Harish-Chandra parameter written

$$\tilde{\pi} \leftrightarrow HC(\tilde{\pi}) = (a_1,\dots,a_{k(+)},-b_{\ell(+)},\dots,-b_1;c_1,\dots,c_{k(-)},-d_{\ell(-)},\dots,-d_1)$$

Here $k(+)+\ell(+) = r$, $k(-)+\ell(-) = s$, and all the a_i, b_i, c_i, d_i are **non-negative** and congruent to $\frac{n-1+m}{2}$ modulo $\mathbb{Z}$. Suppose $n = m$. Then it follows from (2.1.1) that none of the a_i, b_j, etc. equals 0, so the switch from a's to b's, and from c's to d's, is determined uniquely. There is exactly one $\tilde{G}' = \tilde{U}(p,q)$ to which $\tilde{\pi}$ lifts non-trivially under the theta correspondence. This is proved in [L2]; a more general dichotomy result was later proved by Paul in [Pa]. The formula is as follows. We let

$$p = k(+) + \ell(-),\ q = k(-) + \ell(+). \tag{2.1.4}$$

Then the theta lift $\Theta(\tilde{G} \to \tilde{G}'; \tilde{\pi}^\vee)$ to G' of the **contragredient** $\tilde{\pi}^\vee$ of $\tilde{\pi}$ is the non-trivial discrete representation $\tilde{\pi}'$ with Harish-Chandra parameter

(2.1.5)

$$\tilde{\pi}' \leftrightarrow HC(\tilde{\pi}') = (a_1, \dots, a_{k(+)}, -d_{\ell(-)}, \dots, -d_1; c_1, \dots, c_{k(-)}, -b_{\ell(+)}, \dots, -b_1)$$

In other words, the negative parameters (shifted by the difference of signatures) migrate from one part of the signature to the other, and the positive parameters stay put.

On the other hand, when (p, q) does not satisfy (2.1.4), then

$$\Theta(\tilde{G} \to \tilde{U}(p.q); \tilde{\pi}^{\vee}) = 0.$$

When $m = n + 1$, the two choices for the signature (p, q) given by

(2.1.6)

$$p = k(+)+\ell(-)+1,\ q = k(-)+\ell(+);\ p = k(+)+\ell(-),\ q = k(-)+\ell(+)+1$$

both admit non-trivial theta lifts $\tilde{\pi}'$ of $\tilde{\pi}$. In the former case, the Harish-Chandra parameter is given by

(2.1.7)

$$\tilde{\pi}' \leftrightarrow (a_1, \dots, a_{k(+)}, 0, -d_{\ell(-)}, \dots, -d_1; c_1, \dots, c_{k(-)}, -b_{\ell(+)}, \dots, -b_1);$$

whereas in the latter case, the 0 is inserted between the c's and $-b$'s. The reader can check that when m and n are of opposite parity, the Harish-Chandra parameters of both $\tilde{G}$ and $\tilde{G}'$ magically all become **integers**, so the insertion of 0 is consistent.

2.2. **Combinatorics of the correspondence.** The theta correspondence defines an involution θ of the set Π_λ, but if we allow twists by characters of $U(1)$ composed with the determinant, we can use the theta correspondence to define $n + 1$ distinct involutions, as follows. Define a *translate* of $\tilde{\lambda} = (\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_n)$ to be any Langlands parameter of the form

$$\tilde{\lambda}(t) = (\tilde{\lambda}_1 + t > \tilde{\lambda}_2 + t > \cdots > \tilde{\lambda}_n + t)$$

with $t \in \mathbb{Z}$. If $\pi \in \Pi_{\tilde{\lambda}}$ then $\pi \otimes \det^t \in \Pi_{\tilde{\lambda}(t)}$. One obtains the Harish-Chandra parameters of representations in $\Pi_{\tilde{\lambda}(t)}$ from those in $\Pi_{\tilde{\lambda}}$ by adding t to each coefficient.

The crucial fact about the formula (2.1.5) is that it involves $\ell = \ell(+) + \ell(-)$ parameters switching sides. For any $0 \leq \ell \leq n$, and any Langlands parameter $\tilde{\lambda}$, let $T_\ell(\tilde{\lambda})$ be any translate $\tilde{\lambda}(t)$ whose first $n-\ell$ coordinates are positive and whose last ℓ coordinates are negative, and then say t is *of index* ℓ relative to $\tilde{\lambda}$. Since the coordinates of $\tilde{\lambda}$ are all in $\frac{1}{2} + \mathbb{Z}$, there is always at least one t of index ℓ for each ℓ. Define $\theta_\ell : \Pi_{\tilde{\lambda}} \to \Pi_{\tilde{\lambda}}$ by

(2.2.1) $$\theta_\ell(\tilde{\pi}) = \theta(\tilde{\pi} \otimes \det{}^t) \otimes \det{}^{-t}$$

for any $t \in \mathbb{Z}$ of index ℓ. It follows from formula (2.1.5) that the result does not depend on the choice of t of index ℓ.

2.2.2. We identify the elements of $\Pi_{\tilde{\lambda}}$ with $\{\pm 1\}^n$ as follows. We first identify $\{\pm 1\}^n$ with the set $\mathcal{P}(n)$ of subsets $I \subset \{1, 2, \dots, n\}$: I corresponds to the element $(e_1, \dots, e_n) \in \{\pm 1\}^n$ with $e_i = -1$ if and only if $i \in I$. The elements of $\Pi_{\tilde{\lambda}, U(r,s)}$ have been identified above with s-tuples $(\beta_1 > \cdots > \beta_s) = (\lambda_{j_1} > \cdots > \lambda_{j_s})$ chosen from the λ_i. Composing these two identifications, we obtain a bijection of $\Pi_{\tilde{\lambda}}$ with $\{\pm 1\}^n$ in such a way that each $\Pi_{\tilde{\lambda}, U(r,s)}$ corresponds to the vectors in $\{\pm 1\}^n$ with s negative entries.

Proposition 2.2.3. *The involutions θ_ℓ of $\Pi_{\tilde{\lambda}}$ generate a group isomorphic to $\{\pm 1\}^n$ that acts simply transitively on $\Pi_{\tilde{\lambda}}$. The action is the natural multiplication action with respect to the identification defined in (2.2.2).*

Proof. For each ℓ, let $f_\ell \in \{\pm 1\}^n$ be the vector $(e_1, \dots, e_n)$ with $e_i = -1$ if and only if $i > n - \ell$. To define θ_ℓ, we start with $\tilde{\pi} \otimes \det^t \in \Pi_{T_\ell(\tilde{\lambda})}$ for some t of index ℓ. Then

$$HC(\tilde{\pi} \otimes \det^t) = (a_1, \dots, a_{k(+)}, -b_{\ell(+)}, \dots, -b_1; c_1, \dots, c_{k(-)}, -d_{\ell(-)}, \dots, -d_1)$$

with $\ell(+) + \ell(-) = \ell$. In terms of the identification of $\Pi_{T_\ell(\tilde{\lambda})}$ with $\{\pm 1\}^n$, $\tilde{\pi} \otimes \det^t$ corresponds to the $(e_1, \dots, e_n) \in \{\pm 1\}^n$ with $e_i = -1$ if and only if $i \in I$, where I is the set $\{c_1, \dots, c_{k(-)}, -d_{\ell(-)}, \dots, -d_1\}$ viewed as of coordinates of $T_\ell(\tilde{\lambda})$. Since $(-b_{\ell(+)}, \dots, -b_1, -d_{\ell(-)}, \dots, -d_1)$ is a permutation of (the shift by t of) the last ℓ coordinates of $\tilde{\lambda}$, an easy calculation with (2.1.5) now shows that the involution θ_ℓ corresponds to multiplication by f_ℓ. This immediately implies the proposition. $\square$

The global significance of this proposition is discussed in the following section.

2.3. **Automorphic representations of $GL(n)$ and L-packets of unitary groups.** In this section F, F^+, and W are as in §0. In the present section we assume for simplicity that $G = U(W)$ is quasi-split at all finite primes. Let Π be a cuspidal automorphic representation of $GL(n)_F$ that satisfies

Hypotheses 2.3.1. *Writing $\Pi = \Pi_\infty \otimes \Pi_f$, where Π_∞ is an admissible $(\mathfrak{g}, K_\infty)$-module, we have*

(i) *(Regularity) There is a finite-dimensional complex algebraic irreducible representation $W(\Pi) = W_\infty$ of $GL(n, F \otimes_{\mathbb{Q}} \mathbb{R})$ such that*

$$H^*(\mathfrak{g}, K_\infty; \Pi_\infty \otimes W_\infty) \neq 0.$$

(ii) *(Polarization) The contragredient* $\Pi^\vee$ *of* Π *satisfies*

$$\Pi^\vee \xrightarrow{\sim} \Pi \circ c,$$

where c *denotes complex conjugation.*

Then as long as $F^+ \neq \mathbb{Q}$, Labesse [Lab] has proved that Π descends to a stable L-packet Π_G of automorphic cohomological representations of G; if $F^+ = \mathbb{Q}$ the same theorem is proved, but not stated explicitly, by Morel [M, cf. Cor. 9.4.6].

The irreducible representation $W(\Pi)$ factors over the set Σ^+ of real embeddings of F^+

$$W(\Pi) = \otimes_{v \in \Sigma^+} W_v,$$

where W_v is an irreducible representation of

$$GL(n, F \otimes_{F^+,v} \mathbb{C}) \xrightarrow{\sim} GL(n,\mathbb{C}) \times GL(n,\mathbb{C}),$$

each factor associated to a prime (say w, w^c) of F extending v. We write $W_v = W_w \otimes W_{w^c}$, and we always choose w to be the element of our fixed CM type Σ. Each factor is parametrized by its infinitesimal character $\mu(w), \mu(w^c)$. Thus

$$\mu(w) = (\mu_1(w) > \mu_2(w) > \cdots > \mu_n(w))$$

and the polarization condition implies that the two factors are dual, or equivalently that

$$\mu_i(w^c) = -\mu_{n+1-i}(w). \tag{2.3.2}$$

Using w, we identify $Lie(G)_{v,\mathbb{C}} \xrightarrow{\sim} \mathfrak{gl}(n,\mathbb{C})$, and we let $\lambda_v = \mu(w^c)$; let $\Pi_{G,\infty}$ be the discrete series L-packet of $G_\infty = G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ with infinitesimal character λ_v. For each $w \in \Sigma$, with restriction v to F^+, there is a signature $(r_v, s_v) = (r_w, s_w)$ such that $G(F_v^+) \xrightarrow{\sim} U(r_v, s_v)$; then

$$\Pi_{G,\infty} = \prod_{w \in \Sigma} \Pi_{G,w}; \ |\Pi_{G,w}| = \frac{n!}{r_v! s_v!}. \tag{2.3.3}$$

Hypotheses 2.3.4. *The global* L*-packet* Π_G *is expected to satisfy the following properties.*

(a) *Let* π_f *be a representation of* $G(\mathbf{A}_f)$ *such that, for some representation* π_∞ *of* G_∞*,* $\pi_\infty \otimes \pi_f \in \Pi_G$*. Then* $\pi_\infty \in \Pi_{G,\infty}$*. Moreover, for every* $\pi_\infty \in \Pi_{G,\infty}$*, the representation* $\pi_\infty \otimes \pi_f$ *occurs in the discrete automorphic spectrum of* G *with multiplicity one.*

(b) *For all finite* v*, the local component* π_v *of* π_f *is tempered.*

Property (a), specifically the multiplicity one property, is verified under the simplifying hypotheses of [Lab] provided $W(\Pi)$ is sufficiently regular, and property (b) is proved in [HT] provided Π_w is in the discrete series for some finite place w of F split over F^+. Given the recent progress toward completing the theory of endoscopic lifting, it's likely that these properties will be proved completely in the next few years, and we will take them for granted in what follows. They are mentioned here merely in order to provide background for the discussion of motives in the next section.

2.3.1. *L-packets and the theta correspondence.* Let G now vary among the set $\mathcal{G}(n)$ of inner forms of $U(n)$ that are quasi-split at all finite primes. Proposition 2.2.3 indicates that it is possible, to use the theta correspondence, together with twisting by Hecke characters of different archimedean types, to switch archimedean components in the L-packets $\{\Pi_G, G \in \mathcal{G}(n)\}$. Whether or not this switching acts transitively on the archimedean components depends on the non-vanishing of the theta correspondence. This depends on both local and global considerations. The import of Proposition 2.2.3 is that there is no archimedean local obstruction to switching. In some cases non-archimedean local obstructions may require expanding $\mathcal{G}(n)$ to include inner forms that are not quasi-split at all finite primes, in accordance with the dichotomy conjecture of [HKS], but the relation between this conjecture and local L-packets has not been determined (though there is a conjecture due to D. Prasad on this question, discussed below). More serious is the global obstruction, in the form of the possible vanishing of L-functions at the central critical point; we return to this issue in subsequent sections.

3. Rationality questions

3.1. **Motives for automorphic representations of $GL(n)$.** Let Π be a cuspidal automorphic representation of $GL(n)_F$ satisfying Hypotheses 2.3.1. Let $E(\Pi)$ be the field of definition of Π_f; it is a number field (cf. [Cl]), indeed a CM field or a totally real field. It is proved in [CHL2, Shin], [CH], and [So] that to Π satisfying Hypotheses 2.3.1 one can associate a compatible family of continuous representations

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathrm{F}) \to \mathrm{GL}(\mathrm{n}, \mathrm{E}(\Pi)_\lambda),$$

where λ varies over nonarchimedean completions of $E(\Pi)$, satisfying the following identity of local Euler factors almost everywhere:

$$L_v(s - \frac{n-1}{2}, \Pi) = L_v(s, \rho_{\Pi,\ell}), v \notin S \tag{3.1.1}$$

where S is some finite set of places v of F. We postulate the existence of a motive $M(\Pi)$ of rank n over F, pure of weight $n-1$, with coefficients in $E(\Pi)$, such that, for all λ $\rho_{\Pi,\lambda}$ is the λ-adic realization of $M(\Pi)$:

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathrm{F}) \rightarrow \mathrm{Aut}(\mathrm{M}_\ell(\Pi)) \otimes_{\mathrm{E}(\Pi)} \mathrm{E}(\Pi)_\lambda, \tag{3.1.2}$$

where $M_\ell(\Pi)$ is the realization of $M(\Pi)$ on ℓ-adic étale cohomology. When $F^+ \neq \mathbb{Q}$ and n is odd, or when Π_v is in the discrete series for enough finite places v of F^+ (two places generally suffice), $M(\Pi)$ can be constructed as a Grothendieck motive in the cohomology of an abelian scheme over the Shimura variety attached to some unitary (similitude) group.[1] In general one does not even know how to construct a motive in the category of realizations, as considered in Deligne's article on special values of L-functions [D].

The restriction of scalars $R_{F/\mathbb{Q}}M(\Pi)$ is naturally a motive of rank n over $\mathbb{Q}$ with coefficients in $E(\Pi) \otimes F$. The following discussion is a synthesis of the factorization of its period invariants over archimedean places of F, considered in [B], with the construction of quadratic period invariants, carried out in [H2, §1] when $F = \mathbb{Q}$. The de Rham realization of $R_{F/\mathbb{Q}}M(\Pi)$, denoted $M_{F/\mathbb{Q},DR}(\Pi)$, is a free rank n module over $E(\Pi) \otimes F$. The Hodge decomposition

$$M_{F/\mathbb{Q},DR}(\Pi) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{p+q=n-1} M^{p,q}_{F/\mathbb{Q}}(\Pi) \tag{3.1.3}$$

and the natural decomposition of $E(\Pi) \otimes F \otimes \mathbb{C}$-modules

$$M_{F/\mathbb{Q},DR}(\Pi) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{\tau : E(\Pi)\otimes F \ \rightarrow \ \mathbb{C}} M_{F/\mathbb{Q},\tau}(\Pi) \tag{3.1.4}$$

are compatible with the $E(\Pi) \otimes F$-action in the sense that complex conjugation c defines anti-linear isomorphisms

$$c : M^{p,q}_{F/\mathbb{Q},\tau}(\Pi) \xrightarrow{\sim} M^{q,p}_{F/\mathbb{Q},c\tau}(\Pi) \tag{3.1.5}$$

such that

$$c(am) = c(a)c(m), \ a \in E(\Pi) \otimes F, m \in M^{p,q}_{F/\mathbb{Q},\tau}(\Pi). \tag{3.1.6}$$

Here

$$M^{p,q}_{F/\mathbb{Q},\tau}(\Pi) = M^{p,q}_{F/\mathbb{Q}}(\Pi) \cap M_{F/\mathbb{Q},\tau}(\Pi).$$

One expects the following properties to hold

(3.1.7(a)) For all p, q, τ, $\dim M^{p,q}_{F/\mathbb{Q},\tau}(\Pi) \leq 1$.

(3.1.7(b)) For all p, q, $\dim M^{p,q}_{F/\mathbb{Q},\tau}(\Pi)$ is independent of the restriction of τ to $E(\Pi) \otimes 1$.

[1] This is proved when Π_v is in the discrete series for finite places on pp. 98-99 of [HT], although the word "motive" is not used there. For the general case one uses the construction of [Shin] or [CHL2].

(3.1.7(c)) Let τ be as above and denote by $w \in \Sigma_F$ its restriction to $1 \otimes F$, $v \in \Sigma^+$ its restriction to F^+. Let $\mu(w)$ be the infinitesimal character of the finite-dimensional representation W_w (cf. §2.3) and let

$$p(w) = \mu(w) + \frac{n-1}{2}(1, 1, \ldots, 1) := (p_1(w), p_2(w), \ldots, p_n(w))$$

so that for all i, (2.3.2) implies that

$$p_i(w) + p_{n+1-i}(w^c) = n - 1.$$

Then $\dim M^{p,q}_{F/\mathbb{Q},\tau}(\Pi) = 1$ if and only if $(p, q) = (p_i(w), n - 1 - p_i(w^c))$ for some $i \in \mathbf{n} := \{1, \ldots, n\}$.

(3.1.7(d)) The motive $R_{F/\mathbb{Q}}M(\Pi)$ has a non-degenerate polarization

$$< \bullet, \bullet >: R_{F/\mathbb{Q}}M(\Pi) \otimes R_{F/\mathbb{Q}}M(\Pi) \to \mathbb{Q}(1 - n)$$

that is alternating if n is even and symmetric if n is odd. The involution on the coefficients $E(\Pi) \otimes F$ induced by this polarization coincides with complex conjugation. In particular, the polarization induces a non-degenerate hermitian pairing

$$< \bullet, \bullet >_{i,w}: M^{p_i(w),n-1-p_i(w^c)}_{F/\mathbb{Q},\tau}(\Pi) \otimes M^{p_i(w^c),n-1-p_i(w)}_{F/\mathbb{Q},\tau}(\Pi) \to \mathbb{C}$$

for each pair (i, w).

Let $q_i(w) = n - 1 - p_i(w^c)$. For each pair $(i, w) \in \mathbf{n} \times \Sigma_F$, we let $\omega_{i,w}(\Pi) \in M^{p_i(w),q_i(w)}_{F/\mathbb{Q},\tau}(\Pi)$ be the non-zero image of some $\mathbb{Q}$-rational class in $M_{F/\mathbb{Q},DR}(\Pi)$ (cf. [H2, §1.4]. Let

$$Q_{i,w}(\Pi) =< \omega_{i,w}(\Pi), F_\infty(\omega_{i,w}(\Pi)) > \in \mathbb{R}^\times. \tag{3.1.8}$$

Here F_∞ is the action of complex conjugation on the Betti realization of $M_{F/\mathbb{Q},DR}(\Pi)$, cf. [H2, (1.0.4)]. If $\omega_{i,w}(\Pi)$ is given by a differential form Ω on some smooth variety Z underlying the motive $M_{F/\mathbb{Q},DR}(\Pi)$, then $Q_{i,w}(\Pi)$ equals $\int_{Z(\mathbb{C})} \Omega \wedge \star(\Omega)$, with $\star$ the Hodge star operator. This justifies the first claim in

Principle 3.1.9. *For each pair $(i, w) \in \mathbf{n} \times \Sigma_F$, the real number $Q_{i,w}(\Pi)$ is positive. It is well-defined up to a positive element of $\tau(E(\Pi) \otimes F) \cap \mathbb{R}^\times)$.*

The second claim in (3.1.9) follows by considering the behavior of the rational classes under the decomposition (3.1.5), and the fact that the Hodge filtration on $M_{F/\mathbb{Q},DR}(\Pi)$ is rational over F.

The motives $M(\Pi)$ or motives closely related to them can be constructed explicitly in the cohomology of families of abelian varieties over Shimura varieties of PEL type. This is recalled in the next two sections, where it is explained how to identify the quadratic period invariants $Q_{i,w}(\Pi)$ in some cases as integrals of automorphic forms.

3.2. **Periods of automorphic motives and special values of L-functions.** Let $G = U(W)$, with signatures as in §2.3. In this section we assume $F = F^+ \cdot \mathcal{K}$ for some imaginary quadratic field $\mathcal{K}$, and we assume the CM type Σ consists of all complex embeddings of F above a fixed complex embedding of $\mathcal{K}$. To W, or equivalently to G, Shimura has associated a Shimura variety $Sh(W)$ of PEL type (cf. [Ko3]), of dimension $d_W = \sum_{v \in \Sigma^+} r_v s_v$. More precisely, the Shimura variety is associated to a PEL Shimura datum defined for the (rational) similitude group $GU(W)$ of W and to the choice of CM type Σ that identifies $G(F_v^+)$ with $U(r_v, s_v)$ rather than $U(s_v, r_v)$:

$$Sh(W)(\mathbb{C}) = Sh(GU, X(W,\Sigma)) := \varprojlim_{K_f} GU(\mathbb{Q})\backslash X(W,\Sigma) \times GU(\mathbf{A}_f)/K_f.$$

Here $(GU, X(W,\Sigma))$ is the Shimura datum to which $Sh(W)$ is attached and K_f runs through open compact subgroups of $GU(\mathbf{A}_f)$. For the purposes of this exposition the details are not essential.

Let Π be as above. Let $W^+(\Pi)$ be a finite-dimensional (algebraic) representation of GU whose restriction to G is $\otimes_v W_w$, where as above w is the element of Σ extending v. The group $GU(W)(\mathbf{A}_f)$ acts on $Sh(W)$ and on its cohomology groups with coefficients in the local system

$$\tilde{W}^+(\Pi) = \varprojlim_{K_f} GU(\mathbb{Q})\backslash W^+(\Pi) \times X(W,\Sigma) \times GU(\mathbf{A}_f)/K_f$$

attached to $W^+(\Pi)$ of GU.

Let π_f be an irreducible admissible representation of $U(W)(\mathbf{A}_f)$ as in (2.3.4), whose base change is Π_f, and extend π_f to an irreducible admissible representation π_f^+ of $GU(W)(\mathbf{A}_f)$ that occurs as the finite part of an extension π^+ of some element $\pi_\infty \otimes \pi_f \in \Pi_G$, chosen so that π_∞^+ has non-trivial cohomology with respect to $W^+(\Pi)$, see (3.2.8), below. Let

$$M(\pi_f^+) = Hom_{GU(W)(\mathbf{A}_f)}(\pi_f^+, H^{d_W}(Sh(W), \tilde{W}^+(\Pi))). \tag{3.2.1}$$

We are viewing $M(\pi_f^+)$ as a submotive of the (hypothetical) motive corresponding to middle-dimensional cohomology of $Sh(W)$ with coefficients in the local system $\tilde{W}^+(\Pi))$ of geometric origin. For our purposes it suffices to work with the realizations in ℓ-adic, topological, or algebraic de Rham cohomology, in which case one needs to tensor by an appropriate field of coefficients (generally containing $E(\Pi)$. Using ℓ-adic cohomology, we can define the L-function $L(s, M(\pi_f^+))$.

Principle 3.2.2. *There is an explicit Hecke character $\xi_{\pi^+,W}$ of F such that*
$$L(s, M(\pi_f^+)) = L(s - \frac{d_W}{2}, \Pi, \otimes_{w\in\Sigma} \wedge^{s_w} (St), \xi_{\pi^+,W}).$$

As above, we write $(r_w, s_w) = (r_v, s_v)$ if w restricts to v. Cf. [H2, (2.7.9)] for an example of the calculation (dualization and replacement of $\wedge^{r_w}$ by $\wedge^{n-r_w}$) which this shortcuts, . This principle is a theorem in many cases; see the discussion below.

The L-function on the right hand side is the Langlands L-function attached to the indicated representation of the L-group of $R_{F/\mathbb{Q}}(GL(n)_F)$, where St is the identity representation of $GL(n)$, twisted by the Hecke character $\xi_{\pi^+,W}$, which can be calculated explicitly but will here be left unspecified. In the Langlands normalization, the functional equation exchanges s with $1-s$, whereas the motivic L-function on the left hand side has (conjectural) functional equation exchanging s with $1+\omega-s$ where ω is the weight of the motive $M(\pi_f^+)$. This weight is of the form $d_W - k$, where k is an integer (denoted c in [H2]) that depends on the choice of extension π^+ of π, or equivalently on the choice of extension of $W^+(\Pi)$ to a finite-dimensional representation of GU. On the other hand, as in [H2, (2.7.9)], $\xi_{\pi^+,W}^{-1}$ can be attached to a motive $M(\xi_{\pi^+}^{-1})$ of rank one and weight k over F, hence the equality in Principle 3.2.2 can be rewritten

$$L(s - \frac{d_W}{2}, \Pi, \otimes_{w\in\Sigma} \wedge^{s_w} (St)) = L(s, M(\pi_f^+) \otimes M(\xi_{\pi^+,W}^{-1})) \tag{3.2.3}$$

(ignoring problems of coefficients), which is an equality of L-functions with center of symmetry $s = \frac{d_W+1}{2}$.

The operation represented by the notation $\otimes_{v\in\Sigma^+} \wedge^{s_w} (St)$ can be applied to the motive $M_{F/\mathbb{Q}}(\Pi)$, and so (3.2.3) suggests the following isomorphism of motives

$$M(\pi_f^+) \xrightarrow{\sim} \otimes_{w\in\Sigma} \wedge^{s_w}(St) M_{F/\mathbb{Q}}(\Pi) \otimes M(\xi_{\pi^+,W})(t_W)), \tag{3.2.4}$$

where $M(\xi_{\pi^+,W})$ is the dual of the motive $M(\xi_{\pi^+,W}^{-1})$ and the integer in parentheses denotes Tate twist by
$$t_W := \frac{1}{2}\sum_w s_w(s_w - 1),$$
cf. the discussion in [H2, (2.7.9)]. We will not try to make sense of this isomorphism but rather use it to propose a conjectural relation between the quadratic period invariants of the previous section and the Petersson norms of automorphic forms in Π_G.

The starting point is the de Rham version of 3.2.4, which is still hypothetical insofar as the right-hand side of the equality has not been precisely defined:

(3.2.5)
$$M_{DR}(\pi_f^+)_{\mathbb{C}} \xrightarrow{\sim} \otimes_{w\in\Sigma} \wedge^{s_w}(St)M_{F/\mathbb{Q},DR}(\Pi)_{\mathbb{C}} \otimes (M_{DR}(\xi_{\pi^+,W})(t_W))_{\mathbb{C}},$$

For any $J \subset \mathbf{n}$, $w \in \Sigma_F$, let

$$\Omega_{J,w} = \wedge_{j\in J}\omega_{j,w}, \tag{3.2.6}$$

with the $\omega_{j,w}$ as in §3.1. Let $\mathcal{J}(W)$ be the set of functions $J : \Sigma \to \mathcal{P}(n)$ such that the cardinality $|J(w)|$ equals s_w for each $w \in \Sigma$. A basis of the right hand side of (3.2.5) is then given by vectors of the form

$$\Omega_J := \wedge_{w\in\Sigma}\Omega_{J(w),w} \otimes \Omega(\xi_{\pi^+,W}), J \in \mathcal{J}(W). \tag{3.2.7}$$

Here $\Omega(\xi_{\pi^+,W})$ is a basis of the one-dimensional F-vector space

$$M_{DR}(\xi_{\pi^+,W})(t_W).$$

The basis Ω_J, $J \in \mathcal{J}(W)$, is well adapted to the parametrization of the discrete series of G_∞ introduced in §2.2. Let $\lambda^+ = (\lambda_w, w \in \Sigma)$ be the infinitesimal character of the representation $W^+(\Pi)^\vee)$ (dual of the coefficients). Let K_∞ be a fixed maximal compact subgroup of $GU(\mathbb{R})$. For any admissible $(Lie(GU)_{\mathbb{C}}, K_\infty)$ module σ, let

$$H(\sigma,\Pi) = H^{d_W}(Lie(GU)_{\mathbb{C}}, K_\infty; \sigma \otimes W^+(\Pi)). \tag{3.2.8}$$

By Matsushima's formula, we have

(3.2.9)
$$H^{d_W,0}_{DR}(Sh(W), \tilde{W}^+(\Pi))\otimes\mathbb{C} \xrightarrow{\sim} \oplus_{\lambda_\sigma=\lambda^+} H(\sigma,\Pi)\otimes Hom_{GU(\mathbb{R})}(\sigma, \mathcal{A}^0(GU)).$$

Here, to avoid pointless complications, we let $\mathcal{A}^0(GU)$ denote the cusp forms on $GU(\mathbb{Q})\backslash GU(\mathbf{A})$ and let $H^{d_W,0}_{DR}$ denote cuspidal cohomology, for any reasonable definition; σ runs over irreducible admissible $(Lie(GU)_{\mathbb{C}}, K_\infty)$ modules, up to equivalence. Let $Coh(\Pi)$ be the set of equivalence classes of σ for which $H(\sigma,\Pi) \neq 0$; by Wigner's lemma such σ automatically satisfy the condition $\lambda_\sigma = \lambda^+$. Then we can write

(3.2.10)
$$M_{DR}(\pi_f^+)_{\mathbb{C}} = \oplus_{\sigma\in Coh(\Pi)} H(\sigma,\Pi) \otimes Hom_{GU(\mathbf{A})}(\sigma \otimes \pi_f^+, \mathcal{A}^0(GU)).$$

We assume that $Hom_{GU(\mathbf{A})}(\sigma \otimes \pi_f, \mathcal{A}^0(GU)) = 0$ unless σ is in the discrete series (and we can also assume that every occurrence of $\sigma\otimes\pi_f$ in the automorphic forms on GU is cuspidal). Note that the discrete series of G_∞ with fixed infinitesimal character λ is in one-to-one correspondence with the discrete series of $GU(W,\mathbb{R})$ with fixed infinitesimal character extending λ. Thus we can parametrize the discrete series members of $Coh(\Pi)$ as in (2.2.2) by elements of $\prod_{w\in\Sigma}\mathcal{P}(n)$ of cardinality $s_v = s_w$ in the wth place. In other words, the discrete series members of $Coh(\Pi)$ are parametrized by $\mathcal{J}(W)$. On the other hand,

Fact 3.2.11. *For* $\sigma \in Coh(\Pi)$ *in the discrete series,* $\dim H(\sigma,\Pi) = 1$.

Moreover, under property 2.3.4(a), or rather its analogue for GU, we also have

$$\dim Hom_{GU(\mathbf{A})}(\sigma\otimes\pi_f^+, \mathcal{A}^0(GU)) = 1 \tag{3.2.12}$$

for σ as in (3.2.10). Assuming this to be the case we can write

$$M_{DR}(\pi_f^+)_{\mathbb{C}} \xrightarrow{\sim} \oplus_{J\in\mathcal{J}(W)} M_{DR}(\pi_f^+)_J \tag{3.2.13}$$

as a sum of one-dimensional spaces.

Principle 3.2.14. *Under the inverse of the isomorphism (3.2.6) and the decomposition (3.2.11),* Ω_J *is taken to a basis of the one-dimensional space* $M_{DR}(\pi_f^+)_J$.

In the next section we apply these heuristics to analyze the periods of the automorphic motive $M(\pi_f^+)$.

3.2.15. *Validity of Principle 3.2.14.* This principle is a restatement of a form of Langlands' conjecture on the zeta functions of Shimura varieties, applied to $Sh(W)$. The determination of the unramified factors of the zeta function of PEL Shimura varieties such as $Sh(W)$ was explained in two articles of Kottwitz [Ko1, Ko2] assuming certain conjectures, including the fundamental lemma for endoscopy. For unitary groups the relevant conjectures are largely settled, and when $F^+ = \mathbb{Q}$ Kottwitz' program has been completed by S. Morel [M], apart from the determination of the multiplicity one property 2.3.4 (a). The case $F \neq \mathbb{Q}$ has been considered, still at unramified places, in [CHL2], under some simplifying hypotheses. Thus Principle 2.3.4 can be considered established as far as unramified factors are concerned, up to questions of multiplicity. Under a local ramification hypothesis these results are due to Kottwitz and Clozel and have been known for some time. Under the same hypothesis, the coincidence of the local factors at ramified primes was verified by Harris and Taylor, for a special choice of W, again up to multiplicities. This has been extended more recently by Shin in [Shin], to whom we refer for more precise references.

3.3. **Conjectural period relations.** The de Rham cohomology space $M_{DR}(\pi^f)$ can be defined directly in terms of automorphic forms whether or not we have a way to construct the motive $M_{F/\mathbb{Q}}(\Pi)$ to which it is compared in (3.2.6). We temporarily write $S(W,\Sigma)$ to emphasize that the PEL type depends on the choice of Σ. It is known (as a special case of Langlands' conjecture on Shimura varieties) that

$$S(W,\Sigma)^c \xrightarrow{\sim} S(W, c\cdot\Sigma) \tag{3.3.1}$$

where $c\cdot\Sigma = \Sigma_F \setminus \Sigma$. This is just the Shimura variety attached as above to the Shimura datum $(GU, X(W,\Sigma))$ with the conjugate complex structure. Similarly, the representation $W^+(\Pi)$, restricted to $G \subset GU$, is by construction dual to its complex conjugate; thus there is a pairing

$$[\bullet] : W^+(\Pi) \otimes W^{+,c}(\Pi) \to \mathbb{Q}(k) \tag{3.3.2}$$

where $\mathbb{Q}(k)$ is the space of the representation $\nu^{2k} : GU \to GL(1)$, with ν the similitude factor, cf. [H2, 2.6.8].

With this in mind, it is natural to realize the polarization (3.1.8(d)) of the hypothetical $M_{F/\mathbb{Q},DR}(\Pi)$ and the induced polarization of its exterior powers in terms of the natural L_2 pairing of differential forms on $Sh(W)(\mathbb{C})$ with values in $W^+(\Pi)$ with their complex conjugates. Let $Lie(GU)(\mathbb{C}) = Lie(K_\infty)(\mathbb{C}) \oplus \mathfrak{p}$ be the Cartan decomposition. In terms of vector-valued functions

(3.3.3)

$$\phi_i : GU(\mathbb{Q})\backslash GU(\mathbf{A}) \to W^+(\Pi)\otimes \wedge^{d_i}\mathfrak{p},\ i=1,2,\ d_1+d_2 = \dim \mathfrak{p},$$

the L_2 pairing is hermitian and is given by

$$<\phi_1,\phi_2> = \int_{GU(\mathbb{Q})\backslash GU(\mathbf{A})/\mathbb{R}_+^\times} [\phi_1(g)\wedge\bar{\phi}_2(g)]|\nu(g)|^{-2k}. \tag{3.3.4}$$

Consider Faltings' (dual) Bernstein-Gelfand-Gelfand resolution of $\tilde{W}^+(\Pi)$:

(3.3.5)

$$0 \longrightarrow \tilde{W}^+(\Pi) \to \mathcal{K}^0(\tilde{W}^+(\Pi)) \to \mathcal{K}^1(\tilde{W}^+(\Pi)) \to \ \dots \to \mathcal{K}^{d_W}(\tilde{W}^+(\Pi)) \to 0,$$

where for each i,

$$\mathcal{K}^i(\tilde{W}^+(\Pi)) \xrightarrow{\sim} \bigoplus_{J\in\mathcal{J}(W),\ell(J)=i} \mathcal{E}_J \tag{3.3.6}$$

The decomposition (3.2.8) can be reinterpreted in terms of the Hodge decomposition associated to (3.3.4):

(3.3.7)

$$H^{d_W,0}_{DR}(Sh(W),\tilde{W}^+(\Pi))\otimes\mathbb{C} \xrightarrow{\sim} \bigoplus_{J\in\mathcal{J}(W),|J(w)|=s_w} H^{a(J),0}(Sh(W),\mathcal{E}_J)\otimes\mathbb{C}.$$

Here $\ell(J)$ is an index defined in terms of lengths of elements in a Weyl group, cf. [H1, p. 587] and $a(J)$ has the property that $\ell(J)+a(J) = d_W$. In terms of the differential form expression (3.3.2), $\ell(J)$ is the number of holomorphic differentials and $a(J)$ the number of anti-holomorphic differentials. When $Sh(W)$ is not compact, the cohomology group $H^{a(J),0}(Sh(W), \mathcal{E}_J)$ is the cuspidal part of the coherent cohomology obtained from toroidal compactifications and canonical extensions as in [H1, §2.2]; for the purposes of this exposition we may assume it coincides with the space denoted $H_!^{a_J}$ in [*loc. cit.*]. The (refined) Hodge decomposition (3.2.11) is then obtained from the π_f^+-isotypic component of (3.3.4) by identifying

$$(3.3.8) \qquad M_{DR}(\pi_f^+)_J \xrightarrow{\sim} Hom_{GU(\mathbf{A}_f)}(\pi_f^+, H^{a(J),0}(Sh(W), \mathcal{E}_J) \otimes \mathbb{C}).$$

The details of the construction of (3.3.4) and the other steps have been discussed elsewhere, notably in [H1]. The interest of (3.3.7) is that the space $H^{a(J),0}(Sh(W), \mathcal{E}_J)$ has a natural rational structure over a field F_J containing the reflex field of $Sh(W)$ and contained in the Galois closure $\tilde{F}$ of F, cf. [H1]. It is easy to determine the fields F_J explicitly, but for our purposes here it suffices to work over $\tilde{F}$; the refinement is left to a future article (but the impatient reader can supply it in the meantime). It follows that the right-hand side of (3.3.7) has a natural rational structure over the field $\tilde{F}(\Pi) = \tilde{F} \cdot E(\Pi)$. Let $\Omega_J(\Pi)$ denote a $\tilde{F}(\Pi)$-rational generator of the right-hand side of (3.3.7). Let

$$(3.3.9) \qquad Q_J(\Pi) = < \Omega_J(\Pi), \Omega_J(\Pi) >$$

where the pairing on the right is that of (3.3.3). The positive real constant $Q_J(\Pi)$ is well-defined up to multiplication by a totally positive totally real element of $\tilde{F}(\Pi)$.

We define $c(\xi_\pi^+, W) \in \mathbb{R}^\times$ to be the Deligne period of the rank one motive $M(\xi_{\pi^+,W}(t_W))$ attached to the algebraic Hecke character $\xi_{\pi^+,W}$ and the indicated Tate twist, defined up to a factor in the Galois closure $\tilde{F}(\xi_{\pi^+,W})$ of the coefficient field of $\xi_{\pi^+,W}$, as in §8 of [D]. In other words, it is the vector $\Omega(\xi_{\pi^+,W})$ expressed as a multiple of the one-dimensional F-vector space $M_B(\xi_{\pi^+,W})(t_W)$, where M_B denotes Betti cohomology. When F is imaginary quadratic the definition is as in [H2]; in general it is an explicit product of periods of abelian varieties with complex multiplication by F.

The following conjecture is a simultaneous generalization of Shimura's conjectures for the quadratic periods of Hilbert modular forms, proved (up to undetermined algebraic factors) in [H0, Yos], and the conjectures described in §2 of [H2], partially proved in [H3, H5] in many cases. It is naturally based on the conjectural isomorphism of motives (3.2.4).

Conjecture 3.3.10. *For each pair $(i, w) \in \mathbf{n} \times \Sigma_F$ there is a positive constant $Q_{i,w}(\Pi) \in \mathbb{R}^\times$, well determined up to multiplication* *by a totally positive totally real element of* $\tilde{F}(\Pi)$*, such that, for any* $J \in \mathcal{J}(W)$*, the quotient*

$$\frac{Q_J(\Pi)}{\prod_w \prod_{i\in J(w)} Q_{i,w}\cdot c(\xi_\pi^+, W)} \in \tilde{F}(\Pi)^\times. \tag{3.3.11}$$

One can refine the definitions of the invariants $Q_J(\Pi)$ up to $(F_J \cdot E(\Pi))^\times$, ask that the $Q_{i,w}(\Pi)$ to be similarly refined, and then state a conjecture on the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the quotients in (3.3.10). But, for the reasons to be explained in §4.3, it is unlikely that such period relations can be proved for general J by means of the theta correspondence, even when the $Q_{i,w}(\Pi)$ can be defined in terms of automorphic forms. Since no other method is known for studying these invariants, it seems pointless to make a more precise conjecture.

4. Central values of L-functions of unitary groups

4.1. **Applications of local dichotomy.** We have introduced the Eisenstein series $E(\bullet, s, \varphi, \chi)$ in (1.3); the function $\varphi = \varphi(s)$ is a variable element of the induced representation $I_n(s,\chi) = \otimes'_v I_{n,v}(s,\chi_v)$ where $\otimes'$ is restricted tensor product and $I_{n,v}(s,\chi_v)$ is the local degenerate principal series defined in §4 of [HKS]. If v is a finite prime of F^+ that splits in F then $I_{n,v}(0,\chi_v)$ is irreducible; for such v we define $R_n(V_v^+,\chi) = I_{n,v}(0,\chi_v)$. When v is finite but does not split, Kudla and Sweet [KS] show there is a decomposition

$$I_{n,v}(0,\chi) = R_n^+ \oplus R_n^- \tag{4.1.1}$$

with $R_n^\pm = R_n(V_n^\pm,\chi)$ in the notation of [HKS, 4.1 (ii)]. When v is real the decomposition, due to Lee and Zhu [LZ], is more complicated:

$$I_{n,v}(0,\chi) = \oplus_{p+q=n} R_n(V_n^{p,q}). \tag{4.1.2}$$

The notation $V_n^\pm$ in (4.1.1) designates an n-dimensional hermitian space, up to equivalence, with Hasse invariant equal to the sign; in (4.1.1) $V_n^{p,q}$ is the hermitian space over $\mathbb{C}$ with signature (p,q). There is therefore a decomposition

$$I_n(0,\chi) = \oplus_S \otimes_{v\in S} R_n(V_v,\chi) \otimes_{v\notin S} R_n(V_v^+,\chi) \tag{4.1.3}$$

where S runs over *finite* sets of places of F^+ that do not split in F, and V_v is any n-dimensional hermitian space for $v \in S$ (that only finite S are allowed corresponds to the restricted tensor product condition). Thus any φ as above has the property that $\varphi(0) = \sum_S \varphi_S(0)$, with $\varphi_S(0) \in I_{n,v,S}$.

Let $\{V(v)\}$ be a collection of n-dimensional hermitian spaces over F_v, with $V(v) \xrightarrow{\sim} V_n^+$ for all v outside a finite set S of non-split places.

Say $\{V(v)\}$ is *coherent* if there is a global n-dimensional hermitian space V such that $V_v \xrightarrow{\sim} V(v)$ for all v, *incoherent* otherwise. The decomposition (4.1.3) is refined (taking the signatures into account) as a direct sum $I_n(0,\chi) = \oplus_{\{V(v)\}} I_n(\{V(v)\},\chi)$ with the obvious notation.

Consider the theta correspondence between $G' = U(V)$ and $H = U(W \oplus -W)$, defined as in (1.3). There is an isomorphism

$$\Theta_\chi(V_v, 1) \xrightarrow{\sim} R_n(V_v) \tag{4.1.4}$$

where 1 is the trivial representation of V. The isomorphism (4.1.4) induces a surjective map

$$\varphi : \mathcal{S}(V_v) \to R_n(V_v) \tag{4.1.5}$$

where $\mathcal{S}(V_v)$ is the Schwartz-Bruhat space $\mathcal{S}(R_{F_v/F_v^+} W_v \otimes V_v)$.

Theorem 4.1.6. *(Ichino) Let $\varphi(s)$ be a section such that $\varphi(0) \in I_n(\{V(v)\},\chi)$ for a fixed collection as above, with $\varphi(0) = \otimes_v \varphi_v(0)$, $\varphi_v(0) = \varphi(\Phi_v)$ for some $\Phi_v \in \mathcal{S}(V_v)$, $\Phi = \otimes_v \Phi_v$. Then*

(i) *If $\{V(v)\}$ is incoherent, then $E(\bullet, 0, \varphi, \chi) = 0$.*

(ii) *If $\{V(v)\}$ is coherent and attached to the global space V, then*

$$E(\bullet, 0, \varphi, \chi) = \theta_\Phi(\bullet). \tag{4.1.7}$$

in the notation of (1.2.5).

Remark 4.1.8. Part (i) of the above theorem is proved but not stated explicitly in [I2]. [2]

4.2. **Non-negativity of central values.** In this subsection we take $m = n$. Then the choice of a character χ satisfying (1.1.1) determines liftings to the metaplectic group for both $G(\mathbb{A})$ and $G'(\mathbb{A})$. The resulting Weil representation of $G(\mathbb{A}) \times G'(\mathbb{A})$ was denoted $\omega_{V,W,\chi}$ in §1. Here we shall only need to make use the action of $G(\mathbb{A})$, and we will denote it simply by $\omega_\chi = \otimes \omega_{\chi_v}$. Thus the choice of the additive character ψ is also understood implicitly and will not show up in our notations in this subsection.

Recall the *Rallis inner product formula* (1.3.5). Our objective is to show that the central L-value $L(\frac{1}{2}, \pi, St, \chi)$ is non-negative. More

[2] More precisely, Ichino has pointed out that in comparison with the appendix to [HK], which treats the incoherent Eisenstein series on symplectic similitude groups, the unitary group version of Lemma A.1 is contained in (5.1) and Lemma 5.3 (and the references cited there) in [I2]; Lemma A.2 (i) is by definition, Lemma A.2 (ii) is in the proof of [I2, Prop. 6.2], and Lemma A.3 is contained in Lemma 5.1 and Lemma 5.2 of [I2].

generally we will try to show

$$L^S(\frac{1}{2}, \pi, St, \chi) \geq 0$$

for any finite set S of places of F^+. For this let us first re-write (1.3.5) in the pre-normalization form

(4.2.1)

$$< \theta_\phi(f), \theta_{\bar{\phi}}(\bar{f}) >= \prod_{v \in S} Z_v(0, f_v, \bar{f}_v, \varphi_v, \chi_v) \cdot d_n^S(0)^{-1} L^S(\frac{1}{2}, \pi, St, \chi)$$

where, in the notation of [L2], p.182, we have

$$\varphi = \delta(\phi \otimes \bar{\phi}). \tag{4.2.2}$$

Also we are assuming that everything in sight is unramified for $v \notin S$; in particular f_v is a unit vector for $v \notin S$.

In order to avoid any confusion, for the rest of §4.2 we shall use $(\bullet, \bullet)$ to denote various inner products on unitary representations, and $< \bullet, \bullet >$ to denote any invariant bilinear pairing between a representation and its contragredient. As explained in Section 2 of [L2], we have

$$Z_v(0, f_v, \bar{f}_v, \varphi_v, \chi_v) = \int_{G(F_v^+)} (\omega_{\chi_v}(g_v)\phi_v, \phi_v)(\pi_v(g_v)f_v, f_v) dg_v, \tag{4.2.3}$$

provided that the right hand side is absolutely convergent.

Remark 4.2.4. The matrix coefficient $(\omega_{\chi_v}(g_v)\phi_v, \phi_v)$, which occurs in the integrand in (4.2.3), depends only on ω_{χ_v} as an abstract representation in the following sense. Let $\mathcal{F}, \mathcal{F}'$ be any two realizations (models) of the Weil representation ω_{χ_v}. Then there is a unitary intertwining isomorphism $\mathcal{F} \longrightarrow \mathcal{F}'$. If $\phi_v \mapsto \phi'_v$ under this intertwining map then by definition

$$(\omega_{\chi_v}(g_v)\phi_v, \phi_v) = (\omega_{\chi_v}(g_v)\phi'_v, \phi'_v)$$

Thus, in any calculation involving such matrix coefficients, we are free to use any model for ω_{χ_v}, including the well known *Fock models* (when v is archimedean) and *lattice models* when v is non-archimedean.

Instead of using (4.2.1) directly, we shall use a summation of such. In other words, we shall choose suitable local vectors ϕ_v, f_v ($v \in S$), apply (4.2.1), and then take the (finite) summation. The left hand side of the resulting identity is then a sum of non-negative terms. The unramified part, namely

$$d_n^S(0)^{-1} L^S(\frac{1}{2}, \pi, St, \chi)$$

of the right hand side, does not change in this process.

For the right hand side of (4.2.3) to be non-zero, an *a priori* condition is that $\pi_v^\vee$ has a non-zero theta lift to $G'(F_v^+)$. The notion of this local theta lift depends on the choice of χ_v. So we shall speak of the χ_v-theta lift when it is necessary to make this dependence explicit.

The above discussion makes it clear that it will be sufficient to prove the following for every place $v \in S$.

Statement 4.2.5. *Suppose that $\pi_v^\vee$ has a non-zero χ_v-theta lift to $G'(F_v^+)$. There exists finitely many Schwartz functions ϕ_v^i and finitely many vectors f_v^j, such that*

$$0 < \sum_{i,j} \int_{G(F_v^+)} (\omega_{\chi_v}(g_v)\phi_v^i, \phi_v^i)(\pi_v(g_v)f_v^j, f_v^j) dg_v$$

What we can prove so far is

Proposition 4.2.6. *Statement 4.2.5 is valid at least for the following cases:*

(a) *The place v is real and π_v is in the discrete series (which is our standing hypothesis).*

(b) *The place v is finite, and the extension F/F^+ splits at v.*

(c) *The place v is finite, and the representation π_v is tempered.*

The rest of this section is devoted to a proof of this proposition.

4.2.7. *Archimedean places.* Now assume v is archimedean so that $G(F_v^+) = U(p,q)$ for some p, q, and that π_v is in the discrete series. Let $\widehat{U(p,q)}_d$ denote the set of discrete series representations of $U(p,q)$. The construction of [L1], together with the uniqueness result of [Pa] prove that theta correspondence gives rise to a bijection

$$\bigcup_{p+q=n} \widehat{U(p,q)}_d \longleftrightarrow \bigcup_{r+s=n} \widehat{U(r,s)}_d \tag{4.2.7.1}$$

In fact, both [L1] and [Pa] are phrased in terms of the pre-images of the unitary groups in the metaplectic group. Of course, this differs from our χ_v-theta lift by a simple shift of parameters. The explicit χ_v-theta lift has been described in §2.

Assume that $\pi_v^\vee$ has a non-zero χ_v-theta lift to $G'(F_v^+)$, which we denote by π'_v. Then $G'(F_v^+) = U(r,s)$, where the pair (r,s) is uniquely determined by π_v. The representation π'_v is also in the discrete series. We suppose π_v, π'_v are realized on the Hilbert spaces H, H' respectively.

Now the point is that the correspondence (4.2.7.1) is the one described precisely in [L1], which used integrations of matrix coefficients. Let

$$K_v = U(p) \times U(q), \qquad K'_v = U(r,s)$$

be maximal compact subgroups of $G(F_v^+) = U(p,q)$ and $G'(F_v^+) = U(r,s)$, respectively. Let $H_\sigma \subset H$ be the finite dimensional subspace on which K_v acts by σ.

The representation $\pi_v^\vee$ contragredient to π_v is also unitary and belong to the discrete series of $G(F_v^+)$. Suppose it it is realized on the space V. Then $\sigma^\vee$ is the lowest K_v-type of $\pi^\vee$. Let's say it is realized on the subspace $V_{\sigma^\vee}$.

Let $e_1, \cdots, e_n$ be an orthonormal basis for H_σ. Let $f_1, \cdots, f_n \in V_{\sigma^\vee}$ be the dual basis. We may assume that the inner product on V is normalized so that $f_1, \cdots, f_n$ is an orthonormal basis for $V_{\sigma^\vee}$. Let

$$\psi_{\pi_v}(g) = \sum_{j=1}^{n} (\pi_v(g)e_j, e_j) \tag{4.2.7.2}$$

This is our canonical matrix coefficient which differs from that of Flensted-Jensen [FJ] by a factor of $n = \dim \sigma$.

Let $u, v \in H, x, y \in V$. By Schur orthogonality we have

$$\int_{G(F_v^+)} (\pi_v{}^\vee(g)x, y)(\pi_v(g)u, v)dg = \frac{1}{d(\pi_v)} < u, x > \overline{< v, y >}$$

where $<,>$ is the canonical pairing between H and V, and $d(\pi_v)$ is the formal degree of π_v.

As a unitary representation, the Weil representation ω_{χ_v} is realized on some Hilbert space $\mathcal{Y}$. We denote the subspace of smooth vectors by S. (In a Schrödinger model one would have $\mathcal{Y} = L^2(X)$ and $S = S(X)$ for some suitable space X).

Firstly suppose $G'(F_v^+)$ is compact (i.e. $r = 0$ or $s = 0$), so that H' is finite dimensional and we have the embedding

$$H' \otimes V_{\sigma^\vee} \subseteq H' \otimes V_K \subset S$$

where V_K denote the subspace of K_v-finite vectors in $\pi_v^\vee$. If ϕ belongs to the subspace $H' \otimes V_{\sigma^\vee}$ then we may write

$$\phi = \sum_{j=1}^{n} \phi_j \otimes f_j \qquad (\phi_j \in H')$$

so that

$$||\phi||^2 = \sum ||\phi_j||^2$$

For any $u \in H$ we have

$$\int_{G(F_v^+)} (\omega_{\chi_v}(g)\phi, \phi)(\pi_v(g)u, u) = \sum_{i,j} (\phi_i, \phi_j) \int_{G(F_v^+)} (\pi_v^\vee(g)f_i, f_j)(\pi_v(g)u, u)dg$$

$$= \frac{1}{d(\pi_v)} \sum_{i,j} (\phi_i, \phi_j) < u, f_i > \overline{< u, f_j >}$$

That is

$$\int_{G(F_v^+)} (\omega_{\chi_v}(g)\phi, \phi)(\pi_v(g)u, u) = \frac{1}{d(\pi_v)} || \sum_i < u, f_i > \phi_i||^2 \tag{4.2.7.3}$$

Thus positivity is obvious. Obviously the above formula is also valid when $G(F_v^+)$ is compact. It also follows that

(4.2.7.4)

$$\int_{G(F_v^+)} (\omega_{\chi_v}(g)\phi, \phi) \cdot \psi_{\pi_v}(g) dg = \frac{||\phi||^2}{d(\pi_v)} \qquad (G(F_v^+) \,\text{or}\, G'(F_v^+) \,\text{compact})$$

Now return to the general case where neither $G'(F_v^+)$ or $G(F_v^+)$ is necessarily compact. Then at least we have the embedding

$$H' \otimes V_{\sigma^\vee} \subseteq H' \otimes V \subset \mathcal{Y}$$

of Hilbert spaces. Let $\phi \in S$, and let

$$P_\pi : \mathcal{Y} \longrightarrow H' \otimes V_{\sigma^\vee}$$

be the orthogonal projection onto the closed subspace $H' \otimes V_{\sigma^\vee}$. Then we can write

$$P_\pi(\phi) = \sum_{j=1}^n \phi_j \otimes f_j \qquad (\phi_j \in H')$$

By the same calculation as before we obtain the general formula

$$\int_{G(F_v^+)} (\omega_{\chi_v}(g)\phi, \phi) \cdot \psi_{\pi_v}(g) dg = \frac{||P_\pi(\phi)||^2}{d(\pi_v)} \tag{4.2.7.5}$$

Let σ' be the lowest K_v'-type of π_v'. Let $S_{\sigma',\sigma^\vee} \subset S$ be the finite dimensional subspace of joint harmonics of type $\sigma' \otimes \sigma^\vee$ (cf. [Ho]). In [L1] it was shown that the projection operator P_π is non-zero on the subspace $S_{\sigma',\sigma^\vee} \subset S$.

This proves Statement 4.2.5 for our real place v and discrete series representation π_v.

Note that almost exactly the same computation is valid in the non-archimedean case, when π_v is in the discrete series.

4.2.8. *Split (p-adic) places.* Assume that the extension F/F^+ splits at the non-archimedean place v of F^+. Then *any* irreducible admissible unitary representation of $G(F_v^+)$ has a non-zero theta lift to $G'(F_v^+)$. If Z is any vector space over F then there is a natural decomposition

$$Z_v = Z \otimes F_v^+ = Z_v' \oplus Z_v'', \qquad \dim_{F_v^+} Z_v' = \dim_{F_v^+} Z_v'' = \dim_{F^+} Z$$

Applying this to $Z = W \otimes_F V$ we obtain

$$(W \otimes_F V)_v = X_v \oplus Y_v, \qquad \dim_{F_v^+} X_v = \dim_{F_v^+} Y_v = n^2$$

This is a complete polarization with respect to the symplectic form on $(W \otimes_F V)_v$, i.e. both X_v and Y_v are totally isotropic subspaces. Furthermore, we can (and will) identify X_v with $M_n(F_v^+)$, the space of n by n matrices with entries in F_v^+, on which $G(F_v^+) \simeq GL(n, F_V^+)$ and $G'(F_v^+) \simeq GL(n, F_V^+)$ act by pre- and post-multiplications, respectively.

At the place v the Weil representation can be realized on $S(M_n(F_v^+))$, the space of Bruhat-Schartz functions on X_v. The action of $G(F_v^+) \simeq GL(n, F_V^+)$ is then given by

$$\omega_{\chi_v}(g)\phi(x) = \lambda(\det(g))|\det(g)|^{\frac{n}{2}}\phi(xg),$$

where λ is a suitable unitary character of $GL(1, F_v^+)$.

Referring to Remark 4.2.4, we shall use this model to calculate the matrix coefficients

$$(\omega_{\chi_v}(g)\phi_v, \phi_v) \qquad (g \in G(F_v^+))$$

We choose the Schwartz function ϕ_v as follows. Note that $GL(n, F_v^+)$ is an open subset of $M_n(F_v^+)$. We choose an open compact subgroup $K \subset GL(n, F_v^+)$ such that the vector f_v in the space of π_v is fixed by K, and that $\lambda(\det(k)) = 1$ for any $k \in K$. We let ϕ_v be the characteristic function of K. With these choices, it is now clear that

$$(\omega_{\chi_v}(g)\phi_v, \phi_v) = \begin{cases} \text{meas}(K) \cdot (\phi_v, \phi_v), & g \in K \\ 0, & g \notin K \end{cases}$$

Here meas(K) denote the additive measure of K as a subset of $M_n(F_v^+)$. It follows that

$$\int_{G(F_v^+)} (\omega_v(g)\phi_v, \phi_v)(\pi_v(g)f_v, f_v)dg = \text{meas}(K)^2 \cdot (\phi_v, \phi_v)(f_v, f_v)$$

Thus the result is positive. This proves Statement 4.2.5 at the split place v.

It remains to point out that case (c) of Proposition 4.2.6 is a direct consequences of Appendices A and B below. In Theorem A.5 we take

$$G = H = G(F_v^+), \quad \pi_H = \omega_{\chi_v}, \quad \pi_G = \pi_v$$

By the estimate in Theorem 3.2 of [L0], we see that condition (b) of Theorem A.5 is satisfied. On the other hand our representation π_v is assumed to be tempered so it is weakly contained in the regular representation. Thus Theorem A.5 says that our local integral at v is always non-negative. But then Proposition B.4.1 says it is positive for some choice of data. Thus Proposition 4.2.6 is now proved.

The following theorem is an almost immediate consequence:

Theorem 4.2.9. *Let F^+ be a totally real field, F a totally imaginary quadratic extension, $G = U(W)$ a unitary group over F^+ as in §1.3, and and π a cuspidal automorphic representation of G. Let χ be a Hecke character satisfying (1.1.1). Assume*

(a) *For every real place v, π_v is in the discrete series.*

(b) *For every finite place v of F^+ that does not split in F, either v is unramified in F and π_v is unramified, or π_v is tempered.*

Then

$$L(\frac{1}{2}, \pi, St, \chi) \geq 0.$$

Proof. What we have shown so far implies that the product of global terms on the right-hand side of 4.2.3 is positive:

$$d_n^S(0)^{-1} L^S(\frac{1}{2}, \pi, St, \chi) \geq 0$$

In order to complete the proof, we need to show the following:

(1) $d_n(0)^{-1} > 0$;
(2) For each prime $v \in S$, the factor $d_{n,v}(0)^{-1} > 0$;
(3) For each prime $v \in S$, the factor $L_v(\frac{1}{2}, \pi_v, St, \chi_v) > 0$;

Now $d_n(s)$ is a product of Dirichlet series $L(2s + n - r, \eta_{F/F^+}^{n+r})$ with real coefficients, and $s = 0$ is in each case in the closure of the range of absolute convergence of the Euler product. This implies (1). Similarly, each local factor $d_{n,v}(s)$ is an Euler factor with real coefficients, or a Gamma factor, and again $s = 0$ is to the right of all poles; thus (2) follows.

Finally, the finite primes in S are either split in F/F^+ or satisfy hypothesis (b). First assume $v \in S$ is split. Then

$$L_v(s, \pi_v, St, \chi_v) = L_v(s, \pi_v, \chi_v) L_v(s, \pi_v^\vee, \chi_v^{-1}),$$

where π_v is viewed as a representation of $G(F_v^+) \simeq GL(n, F_v^+)$ and the Euler factors on the right-hand side are the principal Euler factors for $GL(n)$. Since π_v is unitary, by hypothesis (b), it follows from the Tadić classification of unitary representations of $GL(n)$ [T] that $L_v(s, \pi_v, St, \chi_v)$ is an Euler factor with real coefficients, and moreover that $s = \frac{1}{2}$ is again to the right of all poles. If now $v \in S$ is not split, we apply hypothesis (b). If π_v is unramified, there was no need to include v in S; thus we may assume π_v tempered. Let w denote the prime of F dividing v. The formal base change Π_w of π_v to $GL(n, F_w)$ then satisfies $\Pi_w^\vee \xrightarrow{\sim} \Pi_w \circ c = \Pi_w$. Since Π_w is tempered, and in particular unitary, it follows that $L_v(s, \pi_v, St, \chi_v) = L(s, \Pi_w, \chi_w)$ is again an Euler factor with real coefficients, and that $s = \frac{1}{2}$ is again to the right of all poles. This proves (3) and thus completes the proof of the theorem. □

4.2.10. *Remarks on the remaining places, and positivity of L-values for $GL(n)$.* The only remaining possibility (for Statement 4.2.5) is the case where v is non-archimedean and F/F^+ remain a non-split quadratic extension at v, and where π_v is possibly non-tempered.

One approach is to carefully choose the local Schwartz functions in a suitable model for the Weil representation. Up to this point we have constructed some rather curious type of matrix coefficients which undoubtedly will play a role in the non-negativity question and beyond. We hope to return to these points soon in an upcoming work.

In any case, by combining the local calculations that have been carried out so far with the main result of [Lab], relating cuspidal cohomological automorphic representations of $GL(n)$ with discrete series L-packets of unitary groups, we obtain

Theorem 4.2.11. *Let F^+ be a totally real field, F a totally imaginary quadratic extension, and Π a cohomological cuspidal automorphic representation of $GL(n, F)$ that satisfies $\Pi^\vee = \Pi^c$. Let χ be a Hekce character satisfying (1.1.1). Assume the extension F/F^+ is unramified at all finite places. Moreover, assume that, if Π_w is ramified at some finite place w of F then w splits over F^+. Then*

$$L(\frac{1}{2}, \Pi, St, \chi) \geq 0.$$

The point is that under the hypotheses on Π and F/F^+, [Lab] shows that Π admits a functorial descent π to any unitary group $G = U(W)$ over F that is quasi-split at all finite places. The functoriality condition implies that

$$L(s, \Pi, St, \chi) = L(s, \pi, St, \chi).$$

The ramification hypotheses then imply that π satisfies conditions (a) and (b) of 4.2.9, and the theorem is a consequence.

Remark 4.2.12. Remarks on functoriality.

As a special case of the generalized Ramanujan conjecture, it is expected that condition (a) in 4.2.9 implies condition (b), cf. 2.3.4. As indicated in §2.3, this can be deduced from results proved in [HT] and [HL] when π is locally a discrete series representation at some finite place v that splits in F/F^+. Indeed, under these hypotheses, π admits a base change to $GL(n, F)$, and after an additional quadratic base change one can reduce to the situation of 4.2.11, where it's no longer important whether or not the local components are tempered where the automorphic representation is ramified. When n is odd, one can similarly apply the results of [Shin]. In the remaining cases, Clozel has recently proved that π_v is tempered at unramified places, but condition (b) is specifically for ramified places.

Dihua Jiang has pointed out to the authors that the result of 4.2.11 can be deduced, without the extraneous ramification hypotheses, by a functoriality argument. Indeed, he has observed that the condition $\Pi^\vee = \Pi^c$ implies that the automorphic induction $\Pi^+ := AI_{F/F^+}(\Pi)$ to an automorphic representation of $GL(2n, F^+)$ is self-dual of symplectic type. Hence Π^+ admits a functorial descent to an automorphic representation π_O of $SO(2n+1, F^+)$. This can be proved either by the L-function method of Ginzburg-Rallis-Soudry or by the twisted trace formula, as in Arthur's forthcoming book. Once one has π_O, the theorem follows from the main result of [LR1]. The proof here is of course quite different.

4.3. **Rationality of the theta lift.** Notation is as in 2.3. In particular, we assume there is a cuspidal automorphic representation Π of $GL(n)$, fixed for the remainder of this section, that descends to a discrete series L-packet on the unitary group G.

We return to the setting of §1 and (3.2). Choose $\phi = \otimes\phi_v \in \mathcal{S}(\mathbb{X}(\mathbf{A}_{F^+}))$ as in (1.2). For all but finitely many places v, the local extension F_v/F_v^+ and the local groups $G(F_v^+)$ and $G'(F_v^+)$ are unramified, so that $\mathbb{W}_v$ contains a self-dual lattice $\mathbb{L}_v$ with integral complete polarization

$$\mathbb{L}_v = \mathbb{L}_v \cap \mathbb{X}_v \oplus \mathbb{L}_v \cap \mathbb{Y}_v$$

and ϕ_v is the characteristic function of $\mathbb{L}_v \cap \mathbb{X}_v$. Let S be the set of places of F^+ where these conditions are not satisfied.

Let $Coh(\Pi)$, π_f, and π_f^+ be as in §3.2; in particular, for any $\sigma \in Coh(\Pi)$,

$$\dim Hom_{GU(\mathbf{A})}(\sigma \otimes \pi_f^+, \mathcal{A}^0(GU)) = 1 \tag{4.3.1}$$

Fix $\pi^+ = \sigma \otimes \pi_f^+$ (we usually refer to π rather than π^+) and let α denote a generator of the one-dimensional space (4.3.1). The representation π_f^+ of $G(\mathbf{A}_f)$ is assumed to have a model over the number field $F(\pi)$ which by [BHR] can be taken to be either a CM field or a totally real field. (Actually we should write $F(\pi_f^+)$ instead of $F(\pi)$, but we ignore this distinction). We may assume α to be $F(\pi)$-rational in the sense of [H2]. In other words, if $\tau \subset \sigma$ is the minimal K-type, the one carrying the cohomology, then, for any vector $v_f \in \pi_f^+(F(\pi))$ and any $F(\pi)$-rational vector $v_\infty \in \tau$, $\alpha(v_\infty \otimes v_f)$ defines an $F(\pi)$-rational coherent cohomology class with respect to a canonical trivialization, cf. [H2], p.113. Similarly, we let $\alpha^\vee$ be a generator of

$$Hom_{GU(\mathbf{A})}((\sigma \otimes \pi_f^+)^\vee, \mathcal{A}^0(GU))$$

which is also one-dimensional and has image in the automorphic forms of coherent cohomological type.

We say ϕ is *algebraic* if in (1.3.3) the term

$$\prod_{v \in S} \tilde{Z}_v(s, f_v, f_v^\flat, \varphi_v, \chi) \in F(\pi, \chi) \tag{4.3.2}$$

whenever s is an integer and $\otimes_v f_v$ and $\otimes_v f_v^\flat$ are the images, under α and $\alpha^\vee$, respectively, of $F(\pi)$ rational vectors in π and $\pi^\vee$.[3] The vector v_∞ is taken to be a fixed non-zero rational element of τ. For example, one can let $\tilde{v}_\infty$ denote a highest weight vector for an appropriate choice of simple roots of the maximal compact subgroup of $G(F_\infty^+)$; $\tilde{v}_\infty$ is then defined over a finite Galois extension F' of $F(\pi, \chi)$, and v_∞ can be taken to be the trace of $\tilde{v}_\infty$ down to $F(\pi, \chi)$. Moreover, φ_v is obtained from $\phi_v \otimes \bar{\phi}_v$ as in (4.3.2), and ϕ_∞ is fixed relative to v_∞ so that the corresponding product of local zeta integrals does not vanish. The condition (4.3.2) is natural at finite primes but ad hoc at archimedean primes, because one should be able to calculate the values of the archimedean local zeta integrals more precisely; however, the condition will suffice for the following discussion.

Algebraicity in the sense just defined is stable under multiplication by a scalar in $F(\pi)$, because $F(\pi)$ is either CM or totally real. We assume the multiplicity one hypothesis (3.2.12) for π and $\pi' := \Theta_\chi(V, \pi)$. Then as f and $f^\flat$ vary among $F(\pi)$-rational vectors and ϕ varies among algebraic Schwartz-Bruhat functions (i.e., ϕ_f varies), the space of $\theta_{\phi\otimes\bar{\phi}}(f)(g')$ defines an $F(\pi, \chi) \cdot \mathbb{Q}^{ab}$-rational structure, say $R_\Theta(\pi')$ on on the $G'(\mathbf{A}_f)$-representation π'_f. Indeed, the action of $G(\mathbf{A}_f) \times G'(\mathbf{A}_f)$ on the Schwartz-Bruhat space stabilizes a $F(\pi) \cdot \mathbb{Q}^{ab}$-rational structure (the coefficients have to include $\mathbb{Q}^{ab}$ because of the dependence on the choice of additive character; cf. the errata to [H3] in [H5]).

On the other hand, π'_f inherits a second $G'(\mathbf{A}_f)$-equivariant $F(\pi)$-rational structure, say $R(\pi')$, from coherent cohomology, as in §3.3.

[3]This is not consistent with the hypotheses made in §1.3 that local and global inner products are compatible. We therefore drop this hypothesis, which was primarily made for convenience in studying the positivity of the central value. The formula (1.3.5) remains true without this compatibility hypothesis, and this is what we are using in the subsequent discussion.

It follows from Schur's Lemma that there is a scalar $c(\pi) \in \mathbb{C}^\times$, well defined up to $F(\pi,\chi)\cdot\mathbb{Q}^{ab,\times}$ such that

$$R_\Theta(\pi') = c(\pi)R(\pi'). \tag{4.3.3}$$

To express the scalars $c(\pi)$ in terms of period invariants, we return to the setting of the earlier sections. In what follows, we use the splitting character χ to identify $\tilde{G} \simeq G$ and $\tilde{G}' \simeq G'$, and likewise identify the Harish-Chandra parameters and infinitesimal characters. Let $\mathcal{J}(\mathbf{n})$ be the set of functions $\mathcal{J} : \Sigma \to \mathcal{P}(n)$, so that

$$\mathcal{J}(\mathbf{n}) = \coprod_W \mathcal{J}(W) \tag{4.3.4}$$

where W runs over all equivalence classes of hermitian spaces over $F \otimes_\mathbb{Q} \mathbb{R}$. In other words, W runs over all collections of signatures $(r_w, s_w; w \in \Sigma)$. If $w \in \Sigma$ lies above $v \in \Sigma^+$, we let λ_w denote the infinitesimal character of the discrete series representation π_v. As in (2.2.2), we can identify

$$\mathcal{J}(\mathbf{n}) = \prod_{w\in\Sigma} \Pi_{\lambda_w} \tag{4.3.5}$$

In this way, the involution θ of §2.2 becomes an involution

$$\theta : \mathcal{J}(\mathbf{n}) \xrightarrow{\sim} \mathcal{J}(\mathbf{n}).$$

Let $J(\pi_\infty) \in \mathcal{J}(W) \subset \mathcal{J}(\mathbf{n})$ correspond to the element π_∞ of the L-packet $\Pi_{G,\infty}$ (notation as in §2.3) under the bijection 4.3.5, and define $J(\pi'_\infty) \in \mathcal{J}(V)$ analogously. It follows from the discussion in §2 that, if $\Theta_\chi(V,\pi) \neq 0$, then

$$\theta(J(\pi_\infty)) = J(\pi'_\infty). \tag{4.3.6}$$

Assume ϕ is an algebraic Schwartz-Bruhat function. The left-hand side of (1.3.5) is then

$$|c(\pi)|^2 \frac{\Omega_{J(\pi'_\infty)}}{\Omega_{J(\pi)}}. \tag{lhs}$$

The right-hand side is, by (4.3.2), an $F(\pi,\chi)$-multiple of

$$d_n(0)^{-1} L(\frac{1}{2}, \pi, St, \chi) \tag{rhs}$$

We now invoke conjectures to rewrite both (lhs) and (rhs) in simpler terms. Conjecture 3.3.10 implies that (lhs) can be rewritten

$$|c(\pi)|^2 \prod_w \frac{\prod_{i'\in J(\pi'_\infty)(w)} Q_{i',w}}{\prod_{i\in J(\pi_\infty)(w)} Q_{i,w}} \cdot \frac{c(\xi^+_{\pi'}, V)}{c(\xi^+_\pi, W)} \tag{4.3.7}$$

A special case of Klingen's theorem on special values of L-functions of Hecke characters of totally real fields implies that $d_n(0)$ is an F-rational multiple of $(2\pi i)^{\frac{dn(n-1)}{2}}$, where $d = [F^+ : \mathbb{Q}]$. To determine the right hand side of (1.3.5) up to algebraic factors, we apply Deligne's conjecture [D] on critical values. This has been worked out in detail in [H2] in terms of the hypothetical invariants $Q_{i,*}$ when $d = 1$. In what follows, when $v \in \Sigma^+$ we write $w(v)$ its extension to an element of the CM type Σ.

Conjecture 4.3.8. *Hypotheses are as above. There is a function* $s_{\Pi,\chi} : \Sigma^+ \rightarrow \{0, \ldots, n\}$ *and an abelian period* $p(s_{\Pi,\chi}) \in \mathbb{C}^\times$*, well defined up to algebraic multiples, so that* $(2\pi i)^{-\frac{d}{n}(n-1)} 2L(\frac{1}{2}, \pi, St, \chi)$ *is an algebraic multiple of*

$$p(s_{\Pi,\chi}) \cdot \prod_{v \in \Sigma^+} \prod_{i=1}^{s_{\Pi,\chi}(w(v))} Q_{i,w(v)}.$$

When $F^+ = \mathbb{Q}$ an exact formula can be found in [H4, **?**], though notation is different; what is here called χ is there called $\alpha^* = \alpha/|\alpha|$, and the χ there is trivial (or rather incorporated into π). In particular, $k = 0$ and κ is the archimedean weight of the product α of χ and an appropriate power of the norm character. The integer $s = s_{\Pi,\chi}(w)$, where w is here the unique real embedding of $\mathbb{Q}$, is determined by the Hodge structure of the motive $M_{F/\mathbb{Q},DR}(\Pi)$. Instead of $\prod_{i=1}^{s} Q_i$ the result is there stated in terms of Petersson norms of the form $Q_J(\Pi)$, defined as in (3.3) in terms of differential forms Ω_J on an appropriate Shimura variety, where J is chosen so that Ω_J is holomorphic. In general, the central character ξ_Π of Π is a Hecke character of F satisfying

$$xi_{\Pi}^{-1} = \xi_\Pi \circ c$$

One can thus view ξ_Π as an automorphic representation of $GL(1)_F$ satisfying Hypotheses 2.3.1. As in §3.1, there is thus an associated motive $M(\xi_\Pi)$ over F of rank one over the coefficient field $E(\xi_\Pi) \subset E(\Pi)$, and for $w \in \Sigma_F$ we define Hodge numbers $(p(w), q(w))$ as in (3.1.7(c)); the subscript i is superfluous. Following 2.4.10 of [H2], we denote them $(\mathcal{P}_w, \mathcal{Q}_w)$. Then for each $w \in \Sigma^+$, we define $s_{\Pi,\chi}(w)$ to be the unique integer in {0, 1, . . . , n} such that

$$\frac{n-\kappa}{2} \leq min(q_{s_{\Pi,\chi}(w)+1} - \kappa - \mathcal{Q}_w, p_{s_{\Pi,\chi}(w)} - \mathcal{P}_w) \tag{4.3.9}$$

One can provide explicit formulas for the abelian period, which incorporates all powers of $2\pi i$ and certain products of periods of the Hecke character α and the central character of Π. For example, in [H4] we have

$$p(s) = \pi^c (2\pi i)^{\frac{n(1-\kappa)}{2} - \frac{n(n-1)}{2} + \kappa s} g(\alpha_0)^s p(\alpha^\vee, 1)^{n-2s}.$$

where $s = s_{\Pi,\alpha^*}(w)$ when w is the complex embedding denoted 1 in [H4]. Comparing the formulas for the two sides of (1.3.5), we thus obtain a conjectural formula for $|c(\pi)|^2$:

Conjecture 4.3.10. *Up to algebraic multiples, we have*

$$|c(\pi)|^2 = p(s_{\Pi,\chi}) \frac{c(\xi_\pi^+, W)}{c(\xi_{\pi'}^+, V)} \prod_{w\in\Sigma} \frac{\prod_{i\in J(\pi_\infty)(w)} Q_{i,w}}{\prod_{i'\in J(\pi'_\infty)(w)} Q_{i',w}} \cdot \prod_{i=1}^{s_{\Pi,\chi}(w)} Q_{i,w}$$

When $d = 1$ this has been proved in certain cases in [H5], up to an undetermined archimedean zeta factor that depends only on Π_∞, and this is used to confirm Deligne's conjecture for these L-functions up to the same undetermined archimedean factor. The proofs in [H5], following [H3], are based on an analysis of the theta correspondence via seesaw pairs and properties of coherent cohomology. But even when $d = 1$ it seems to be completely impossible to prove this conjecture in general using automorphic techniques. In the cases treated in [H3, H5], the representation π' is always of holomorphic type, all the $Q_{i,w}$'s on the right-hand side of the formula in Conjecture 4.3.10 cancel and one finds that $|c(\pi)|^2$ is a product of abelian periods. These are products that arise in the intermediate steps of the rationality testing of the theta lifts, and occur as periods of automorphic forms on smaller groups. In general, however, there is no reason for the $Q_{i,w}$'s to cancel, so the conjecture implies that the transcendental part of $c(\pi)$ involves some non-trivial product of $Q_{i,w}$'s. But these are periods of automorphic representations of $GL(n)$ and cannot be expected to occur in theta correspondences on unitary groups of size less than n. It thus appears that Conjecture 4.3.10 in general can only be obtained as a consequence of Deligne's conjecture, and not vice versa.

Appendix A. Nonnegativity of integrals of certain positive definite functions

Let H be a locally compact Hausdorff topological group, with a fixed left invariant Haar measure μ_H^l. Denote by Δ_H its modula function, which is defined by

$$\int_H f(xh^{-1})\, d\mu_H^l(x) = \Delta_H(h) \int_H f(x)\, d\mu_H^l(x), \quad \text{for all } f \in \mathrm{C}_c(H),\ h \in H.$$

Here and henceforth, "C_c" stands for the space of compactly supported (complex valued) continuous functions. Then

$$\mu_H^r := \Delta_H^{-1}\mu_H^l$$

is a right invariant Haar measure. Set

$$\mu_H^0 := \Delta_H^{-\frac{1}{2}}\mu_H^l = \Delta_H^{\frac{1}{2}}\mu_H^r.$$

Then

$$\int_H f(x^{-1})\, d\mu_H^0(x) = \int_H f(x)\, d\mu_H^0(x), \quad \text{for all } f \in \mathrm{C}_c(H). \tag{A.1}$$

Let ϕ be a continuous complex-valued function on H. Recall that ϕ is said to be positive definite, or equivalently, of positive type, if

$$\sum_{i=1}^n \sum_{j=1}^n c_i\overline{c_j}\phi(h_j^{-1}h_i) \geq 0,$$

for all $n \geq 1$, $c_1, c_2, \cdots, c_n$ in $\mathbb{C}$, and $h_1, h_2, \cdots, h_n$ in H. By the GNS construction (named after Gelfand, Naimark and Segal), ϕ is positive definite if and only if

$$\phi(h) = \langle \pi_H(h)(\xi), \xi\rangle, \quad h \in H, \tag{A.2}$$

for some vector ξ in a unitary representation π_H of H [BHV, Theorem C.4.3 and Theorem C.4.10].

Assume that ϕ is positive definite and integrable (with respect to the measure μ_H^0). It follows from (A.1) and (A.2) that

$$\int_H \phi(x)\, d\mu_H^0(x) \in \mathbb{R}. \tag{A.3}$$

The inequality

$$\int_H \phi(x)\, d\mu_H^0(x) \geq 0 \tag{A.4}$$

holds in many cases, but not in general. The purpose of this note is to prove (A.4) in a special case which is interesting in representation theory and automorphic forms.

Let G be a locally compact Hausdorff topological group, with a compact subgroup K and a closed subgroup P such that $G = KP$.

Denote by Δ_G and Δ_P the modular function of G and P, respectively. Write

$$\delta(p) = \frac{\Delta_G(p)}{\Delta_P(p)}, \quad p \in P,$$

and extend it to a left K-invariant function on G, which is still denoted by δ, by the formula

$$\delta(kp) = \delta(p), \quad k \in K,\, p \in P.$$

Under this general setting, Harish-Chandra's basic spherical function Ξ is still defined by

$$\Xi(g) = \Xi_{K,P}(g) = \int_K \delta^{-\frac{1}{2}}(gk)\, dk, \quad g \in G,$$

where "dk" is the normalized Haar measure on K.

The following is our main result in this appendix, which generalizes [He, Theorem 2.1].

Theorem A.5. *Let π_G be a unitary representation of G which is weakly contained in the regular representation. Let H be a closed subgroup of G and let π_H be a unitary representation of H. Let*

$$u_\pi = \sum_{i=1}^{k} u_i \otimes v_i \in \pi := \pi_H \widehat{\otimes} (\pi_G|_H).$$

Assume that

(a) *$v_1, v_2, \cdots, v_k$ are all K-finite,*
(b) *and the function $\langle \pi_H(h) u_i, u_j\rangle\, \Xi(h) \in \mathrm{L}^1(H; \mu_H^0)$, $i, j = 1, 2, \cdots, k$.*

Then the function

$$\langle \pi(h) u_\pi, u_\pi\rangle \in \mathrm{L}^1(H; \mu_H^0), \tag{A.6}$$

and its integral

$$\int_H \langle \pi(h) u_\pi, u_\pi\rangle\, d\mu_H^0(h) \geq 0. \tag{A.7}$$

Let π_1 and π_2 be two unitary representations of G. Recall that π_1 is said to be weakly contained in π_2 if for all $v \in \pi_1$, the matrix coefficient $\langle \pi_1(g)v, v\rangle$ is contained in the closure (with respect to the topology of uniform convergence on compacta) of the space spanned by $\{\langle \pi_2(g)v', v'\rangle \mid v' \in \pi_2\}$.

Theorem A.5 is easily reduced to the case when G is σ-compact. Therefore, without lose of generality, we assume in the remaining part of this note that G is σ-compact.

The following lemma is elementary and known ([BHV, Page 421, Remark F.1.2 (ix)]).

Lemma A.8. *Assume that π_1 is weakly contained in π_2. Let $v_1, v_2, \cdots, v_k \in \pi_1$. Then there is a family of vectors*

$$\{u_{i,r,n}\}_{1 \leq i \leq k, 1 \leq r, n < \infty}$$

in π_2 *such that*

$$\lim_{n\to\infty}\sum_{r=1}^{\infty}\langle \pi_2(g)u_{i,r,n}, u_{j,r,n}\rangle = \langle \pi_1(g)v_i, v_j\rangle, \quad i,j = 1,2,\cdots,k,$$

uniformly on compact subsets of G.

Denote by $\widehat{K}$ the set of equivalence classes of finite dimensional irreducible unitary representations of K. For every $\tau \in \widehat{K}$, write $\pi_1(\tau)$ for the τ-isotypic component of V, which is automatically a closed subspace of π_1. Let S be a finite subset of $\widehat{K}$. Set

$$\pi_1(S) = \oplus_{\tau\in S}\pi_1(\tau),$$

and

$$d_S = \sum_{\tau\in S}\deg(\tau)^2.$$

A vector in $\pi_1(S)$ is said to be of type S.

Recall that the left and right regular representations are canonically isomorphic (so there is no confusion when we refer to the regular representation in Theorem A.5). We work with the right regular representation. It is realized on the Hilbert space $\mathrm{L}^2(G;\mu_G^r)$, and the action is given by the right translation R.

Lemma A.9. *Let* π_G *and* $v_1, v_2, \cdots, v_k \in \pi_G$ *be as in Theorem A.5. Assume that all* $v_1, v_2, \cdots, v_k$ *are of type* S. *Then there is a family of vectors*

$$\{f_{i,r,n}\}_{1\leq i\leq k, 1\leq r,n<\infty}$$

in $\mathrm{C}_c(G) \subset \mathrm{L}^2(G;\mu_G^r)$, *all have type* S, *such that*

$$\lim_{n\to\infty}\sum_{r=1}^{\infty}\langle R_g f_{i,r,n}, f_{j,r,n}\rangle = \langle \pi_G(g)v_i, v_j\rangle, \quad i,j = 1,2,\cdots,k, \tag{A.10}$$

uniformly on compact subsets of G.

Proof. This is a consequence of Lemma A.8 by applying the orthogonal projection map

$$\mathrm{L}^2(G;\mu_G^r) \to \mathrm{L}^2(G;\mu_G^r)(S).$$

□

The following lemma is a formal generalization of a fundamental result of Cowling, Haagerup and Howe [CHH, Theorem 2]. Although in [CHH], the theorem is only stated for G semisimple algebraic, and $G = KP$ an Iwasawa decomposition, the same proof works in our general setting. So we omit the proof.

Lemma A.11. *Assume that π_G is a unitary representation of G which is weakly contained in the regular representation. Then*

$$|\langle \pi(g)u, v\rangle| \leq d_S||u||\,||v||\,\Xi(g),$$

for all $u, v \in \pi_G(S)$ and $g \in G$.

Now(A.6) is a direct consequence of Lemma A.11.

Fix a family $\{f_{i,r,n}\}$ as in Lemma A.9. Take $i = j$ and $g = 1$ in (A.12), we get

$$\lim_{n\to\infty}\sum_{r=1}^{\infty}||f_{i,r,n}||^2 = ||v_i||^2, \quad i = 1, 2, \cdots, k. \tag{A.12}$$

By Lemma A.11, we have that

$$|\sum_{r=1}^{\infty}\langle R_g f_{i,r,n}, f_{j,r,n}\rangle| \leq \sum_{r=1}^{\infty} d_S||f_{i,r,n}||\;||f_{j,r,n}||\,\Xi(g) \leq \frac{d_S}{2}(||v_i||^2+||v_j||^2)\Xi(g).$$

Now Lebesgue's dominated convergence theorem applies and we have

$$\begin{aligned}
&\int_H \langle \pi(h)u_\pi, u_\pi\rangle\, d\mu_H^0(h)\\
=&\int_H \sum_{i,j}\langle \pi_H(h)u_i, u_j\rangle\,\langle \pi_G(h)v_i, v_j\rangle\, d\mu_H^0(h)\\
=&\sum_{i,j}\lim_{n\to\infty}\int_H \langle \pi_H(h)u_i, u_j\rangle\,\sum_{r=1}^{\infty}\langle R_h f_{i,r,n}, f_{j,r,n}\rangle\, d\mu_H^0(h)\\
=&\lim_{n\to\infty}\sum_{r=1}^{\infty}\int_H \sum_{i,j}\langle \pi_H(h)u_i, u_j\rangle\,\langle R_h f_{i,r,n}, f_{j,r,n}\rangle\, d\mu_H^0(h)
\end{aligned}$$

Therefore, to prove (A.7), it suffices to show that for all $f_1, f_2, \cdots, f_k \in \mathrm{C}_c(G)$, we have

$$\int_H \sum_{i,j}\langle \pi_H(h)u_i, u_j\rangle\,\langle R_h f_i, f_j\rangle\, d\mu_H^0(h) \geq 0,$$

or, equivalently,

$$\int_G\int_H \sum_{i,j}\langle \pi_H(h)u_i, u_j\rangle\, f_i(gh)\overline{f_j(g)}\, d\mu_H^0(h)\, d\mu_G^r(g) \geq 0, \tag{A.13}$$

Lemma A.14. *Let $f \in \mathrm{C}_c(G)$. If for all $g \in G$,*

$$\int_H f(gh)\frac{\Delta_G(h)}{\Delta_H(h)}\, d\mu_H^l(h) \geq 0, \tag{A.15}$$

then

$$\int_G f(x)\, d\mu_G^l(x) \geq 0,$$

Proof. Let ρ be a positive-valued continuous function on G such that

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x), \quad x \in G,\, h \in H. \tag{A.16}$$

Such a function always exists [R, Chapter 8, Section 1]. There is a positive Borel measure μ on G/H such that (see [BHV, Theorem B.1.4])

$$\int_G f(x)\, d\mu_G^l(x) = \int_{G/H}\int_H f(xh)\rho^{-1}(xh)\, d\mu_H^l(h)\, d\mu(x).$$

Now the lemma follows from (A.16) and (A.15). □

Fix an arbitrary $g \in G$. By Lemma A.14, (A.13) is implied by

$$\int_H\int_H \sum_{i,j}\langle \pi_H(h)u_i, u_j\rangle\, f_i(gxh)\overline{f_j(gx)}\, d\mu_H^0(h)\, \Delta_G^{-1}(gx)\frac{\Delta_G(x)}{\Delta_H(x)}\, d\mu_H^l(x) \geq 0,$$

or equivalently,

$$\int_H\int_H \sum_{i,j}\langle \pi_H(xh)u_i, \pi_H(x)u_j\rangle\, f_i(gxh)\overline{f_j(gx)}\, \Delta_H^{-\frac{1}{2}}(xh)\, \Delta_H^{-\frac{1}{2}}(x)\, d\mu_H^l(h)\, d\mu_H^l(x) \geq 0.$$

By changing of variable, this is the same as

$$\int_H\int_H \sum_{i,j}\langle \pi_H(h)u_i, \pi_H(x)u_j\rangle\, f_i(gh)\overline{f_j(gx)}\, \Delta_H^{-\frac{1}{2}}(h)\, \Delta_H^{-\frac{1}{2}}(x)\, d\mu_H^l(h)\, d\mu_H^l(x) \geq 0,$$

The left hand side of the above equality equals to

$$\langle \int_H \sum_i f_i(gh)\pi_H(h)u_i, \int_H \sum_j f_j(gx)\pi_H(x)u_j\rangle,$$

which is obviously nonnegative. This finishes the proof of (A.7).

Appendix B. Matrix coefficient integrals and theta correspondence

B.1. **Oscillator representations of Jacobi groups.** Let k_0 be a fixed local field of characteristic zero, and let k/k_0 be a quadratic extension, with "–" the nontrivial Galois element. For global applications, the split case of $\mathrm{k} = \mathrm{k}_0 \times \mathrm{k}_0$ is included.

Fix $\epsilon = \pm 1$. Let E be an ϵ-hermitian k-module, i.e., it is a free k-module of finite rank, equipped with a non-degenerate k_0-bilinear map

$$\langle\, ,\, \rangle : E \times E \to \mathrm{k}$$

such that

$$\langle u,v\rangle = \epsilon\overline{\langle v,u\rangle}, \quad \langle au,v\rangle = a\langle u,v\rangle, \quad a\in \mathrm{k},\, u,v\in E.$$

Similarly, let E' be an ϵ-skew hermitian k-module. Then

$$\mathbf{E} := E\otimes_{\mathrm{k}} E'$$

is a skew hermitian module under the form

$$\langle u\otimes u', v\otimes v'\rangle := \langle u,v\rangle\cdot\langle u',v'\rangle.$$

Denote by

$$\mathrm{H}(\mathbf{E}) := \mathbf{E}\times \mathrm{k}_0$$

the associated Heisenberg group, whose multiplication is given by

$$(u,t)\,(u',t') := (u+u', t+t'+\frac{1}{2}\mathrm{tr}_{\mathrm{k}/\mathrm{k}_0}\langle u,u'\rangle).$$

The unitary group (or general linear group in the split case) $\mathrm{U}(E)$ act on $\mathrm{H}(\mathbf{E})$ as group automorphisms through its action on $\mathbf{E}$. We define the corresponding Jacobi group to be the semidirect product

$$\mathrm{J}_{E'}(E) := \mathrm{U}(E)\ltimes \mathrm{H}(\mathbf{E}).$$

Fix a nontrivial unitary character $\psi : \mathrm{k}\to\mathbb{C}$. By a unitary oscillator representation of $\mathrm{J}_{E'}(E)$, we mean a unitary representation of $\mathrm{J}_{E'}(E)$ whose restriction to $\mathrm{H}(\mathbf{E})$ is irreducible of central character ψ. The underlying smooth representation of a unitary oscillator representation is called a smooth oscillator representation. Write

$$\begin{aligned}\Omega_{E'}(E) &:= \{\text{equivalence classes of unitary oscillator representations of } \mathrm{J}_{E'}(E)\}\\ &= \{\text{equivalence classes of smooth oscillator representations of } \mathrm{J}_{E'}(E)\}.\end{aligned}$$

The group of unitary characters on $\mathrm{U}(E)$ acts simply transitively on $\Omega_{E'}(E)$ by tensor product. Since every character of $\mathrm{U}(E)$ factor through the determinant map, we get an action of the group $\widehat{\mathrm{k}^1}$ of unitary characters of

$$\mathrm{k}^1 = \{x\in \mathrm{k}\mid x\bar{x}=1\}$$

on the set $\Omega_{E'}(E)$.

B.2. **Kudla characters of oscillator representations.** Assume that $n := \mathrm{rank}_{\mathrm{k}} E > 0$.

Let $\widetilde{E}$ be a nonzero split ϵ-hermitian k-module. For any maximal isotropic free submodule X of E,

$$\mathbf{X} := X\otimes_{\mathrm{k}} E'$$

is a maximal isotropic free submodule of

$$\widetilde{\mathbf{E}} := \widetilde{E}\otimes_{\mathrm{k}} E',$$

and is also a subgroup of the Heisenberg group $\mathrm{H}(\widetilde{\mathbf{E}})$. Let $\widetilde{\omega} \in \Omega_{E'}(\widetilde{E})$, and view it a smooth oscillator representation. Then up to scalar, there is a unique nonzero continuous linear functional

$$\lambda_X : \widetilde{\omega} \to \mathbb{C}$$

which is $\mathbf{X}$-invariant. Denote by $\mathrm{P}_X(\widetilde{E})$ the parabolic subgroup of $\mathrm{U}(\widetilde{E})$ preserving X. Since it normalizes $\mathbf{X}$, there is a unique character

$$\kappa_{\widetilde{\omega},X} : \mathrm{P}_X(\widetilde{E}) \to \mathbb{C}^\times$$

such that

$$\lambda_X \in \mathrm{Hom}_{\mathrm{P}_X(\widetilde{E})}(\widetilde{\omega}, \kappa_{\widetilde{\omega},X}).$$

We conclude from [HKS] that $\kappa_{\widetilde{\omega},X}$ is uniquely of the form

$$\kappa_{\widetilde{\omega},X}(p) = |\det(p|_X)|^{\frac{n'}{2}} \kappa_{\widetilde{\omega}}(\det(p|_X)), \quad p \in \mathrm{P}_X(\widetilde{E}),$$

where $n' = \mathrm{rank}_{\mathrm{k}} E'$, and $\kappa_{\widetilde{\omega}} \in \widehat{\mathrm{k}^\times}(n')$. Here

$$\widehat{\mathrm{k}^\times}(n') := \{\chi \in \widehat{\mathrm{k}^\times} \mid \chi|_{\mathrm{k}_0^\times} = \epsilon_{\mathrm{k}/\mathrm{k}_0}^{n'}\},$$

and $\epsilon_{\mathrm{k}/\mathrm{k}_0}$ is the quadratic character of $\mathrm{k}_0^\times$ whose kernel is

$$\{x\bar{x} \mid x \in \mathrm{k}^\times\}.$$

We call $\kappa_{\widetilde{\omega}}$ the Kudla character of $\widetilde{\omega}$. It is independent of X.

We use the doubling method to define Kudla characters in general. Write $E_- := E$ as a k-module, equipped with the ϵ-hermitian form which is negative to the one on E. Assume that

$$\widetilde{E} := E \oplus E_-.$$

Then $\mathrm{J}_{E'}(E)$ and $\mathrm{J}_{E'}(E_-)$ are commuting subgroups of $\mathrm{J}_{E'}(\widetilde{E})$.

Let $\omega \in \Omega_{E'}(E)$, there is a unique $\widetilde{\omega} \in \Omega_{E'}(\widetilde{E})$ such that as unitary representations of $\mathrm{J}_{E'}(E) \times \mathrm{J}_{E'}(E_-)$,

$$\widetilde{\omega} \cong \omega \widehat{\otimes} \omega_-$$

for some $\omega_- \in \Omega_{E'}(\widetilde{E}_-)$. We define the Kudla character

$$\kappa_\omega := \kappa_{\widetilde{\omega}}.$$

When E already splits, this definition is consistent with the previous one. Clearly

$$\kappa_{\omega_-} = \kappa_\omega.$$

For every $\chi \in \widehat{\mathrm{k}^1}$, we have that

$$\kappa_{\chi \otimes \omega}(x) = \chi\left(\frac{x}{\bar{x}}\right) \kappa_\omega(x).$$

Kudla characters establish a one-one correspondence

$$\Omega_{E'}(E) \leftrightarrow \widehat{\mathrm{k}^\times}(n').$$

B.3. **Degenerate principal series.** We continue with the notation of the last section. Assume that

$$X := \{(v, v) \in \widetilde{E} \mid v \in E\}.$$

Let $\widetilde{\omega} \cong \omega \widehat{\otimes} \omega_-$ and λ_X be as before. We have a group isomorphism

$$\mathrm{J}_{E'}(E) \to \mathrm{J}_{E'}(E_-), \quad (x, u, t) \mapsto (x, u, -t).$$

Formally write

$$\bar{\omega} := \{\bar{v} \mid v \in \omega\},$$

viewed as a complex vector space by

$$c\bar{v} := \overline{\bar{c}v}, \quad c \in \mathbb{C},\ v \in \omega.$$

As usual, it is a representation of $\mathrm{J}_{E'}(E)$ which is contragredient to ω. Denote by $\kappa_{\omega,\mathrm{J}}$ the character of $\mathrm{J}_{E'}(E)$ given by

$$(x, u, t) \mapsto \kappa_\omega(\det(x)).$$

It is important to observe that as representations of $\mathrm{J}_{E'}(E)$,

$$\omega_- \cong \bar{\omega} \otimes \kappa_{\omega,\mathrm{J}},$$

and the linear functional λ_X can be identified (up to scalar) with

$$\widetilde{\omega} \cong \omega \widehat{\otimes} \bar{\omega} \otimes \kappa_{\omega,\mathrm{J}} \to \mathbb{C}, \quad u \otimes \bar{v} \otimes 1 \mapsto \langle u, v \rangle.$$

Recall that

$$\lambda_X \in \mathrm{Hom}_{\mathrm{P}_X(\widetilde{E})}(\widetilde{\omega}, \kappa_{\widetilde{\omega},X}),$$

and

$$\kappa_{\widetilde{\omega},X}(p) = |\det(p|_X)|^{\frac{n'}{2}}\, \kappa_\omega(\det(p|_X)), \quad p \in \mathrm{P}_X(\widetilde{E}).$$

Frobenius reciprocity produces a $\mathrm{U}(\widetilde{E})$-intertwining continuous linear map

$$\Lambda_X : \widetilde{\omega} \to \mathrm{I}(\kappa_\omega, s_0), \quad \text{with } s_0 = \frac{n' - n}{2}.$$

Here for every character χ of $\mathrm{k}^\times$, and every $s \in \mathbb{C}$, $\mathrm{I}(\chi, s)$ is the normalized induction consisting of all smooth functions f on $\mathrm{U}(\widetilde{E})$ such that

$$f(pg) = \chi(\det(p|_X))\, |\det(p|_X)|^{s+\frac{n}{2}} f(g), \quad p \in P_X, g \in \mathrm{U}(\widetilde{E}).$$

Note that the diagram

$$\text{(B.3.1)} \qquad \begin{array}{ccc} \omega \otimes \bar{\omega} & \xrightarrow{u\otimes \bar{v} \mapsto u \otimes \bar{v} \otimes 1} & \omega \widehat{\otimes} (\bar{\omega} \otimes \kappa_{\omega,\mathrm{J}}) \cong \widetilde{\omega} \\ {\scriptstyle c^{\omega}} \downarrow & & \downarrow {\scriptstyle \Lambda_X} \\ \mathrm{C}^{\infty}(\mathrm{U}(E)) & \xleftarrow{\text{the restriction map}} & \mathrm{I}(\kappa, s_0) \end{array}$$

commutes, where c^{ω} is the matrix coefficient map given by

$$c^{\omega}_{u\otimes\bar{v}}(g) := \langle \omega(g)u, v \rangle.$$

Proposition B.3.2. *Fix* $\kappa_0 \in \widehat{\mathrm{k}^{\times}}(n)$. *Let* $E'_1, E'_2, \cdots, E'_k$ *be representatives of equivalent classes of* ϵ*-skew hermitian* k*-modules of rank* n. *Let* $\widetilde{\omega}_i \in \Omega_{E'_i}(\widetilde{E})$, *with Kudla character* κ_0. *Then* $\Lambda_X(\widetilde{\omega}_i)$, $i = 1, 2, \cdots, k$ *are pairwise inequivalent irreducible subrepresentations of* $\mathrm{I}(\kappa_0, 0)$, *and*

$$\mathrm{I}(\kappa_0, 0) = \bigoplus_{i=1}^{k} \Lambda_X(\widetilde{\omega}_i).$$

Note that $k = 1$ if k splits, $k = 2$ if k is a nonarchimedean local field, and $k = n + 1$ if $\mathrm{k} = \mathbb{C}$.

The following is a direct consequence of (B.3.1) and Proposition B.3.2.

Corollary B.3.3. *Fix* $\kappa_0 \in \widehat{\mathrm{k}^{\times}}(n)$. *Let* $E'_1, E'_2, \cdots, E'_k$ *be representatives of equivalent classes of* ϵ*-skew hermitian* k*-modules of dimension* n. *Let* $\omega_i \in \Omega_{E'_i}(E)$, *with Kudla character* κ_0. *Then*

$$\bigoplus_{i=1}^{k} c^{\omega_i}(\omega_i \otimes \bar{\omega}_i)$$

is dense in $\mathrm{C}^{\infty}(\mathrm{U}(E))$.

B.4. **Theta correspondences and matrix coefficient integrals.** Assume that $n = n'$. Let ω be a smooth oscillator representation of $\mathrm{J}_{E'}(E)$ as before. Let π be the underlying smooth representation of a tempered irreducible unitary representation of $\mathrm{U}(E)$. By [Su, Theorem 1.2] and the method of [L0, Theorem 3.2], we know that

$$\begin{array}{rcl} \omega \widehat{\otimes} \bar{\omega} \widehat{\otimes} \pi \widehat{\otimes} \bar{\pi} & \to & \mathrm{L}^1(\mathrm{U}(E)), \\ u \otimes \bar{v} \otimes u' \otimes \bar{v'} & \mapsto & (g \mapsto \langle \omega(g)u, v \rangle \, \langle \pi(g)u', v' \rangle) \end{array}$$

is a well defined continuous linear map.

Proposition B.4.1. *The space* $\mathrm{Hom}_{\mathrm{U}(E)}(\omega\widehat{\otimes}\pi, \mathbb{C})$ *is nonzero if and only if*

$$\int_{\mathrm{U}(E)} \langle \omega(g)u, u\rangle \, \langle \pi(g)v, v\rangle \, dg \neq 0 \tag{B.4.2}$$

for some $u \in \omega$ *and* $v \in \pi$.

Proof. The "if" part is trivial. Now assume that all matrix coefficient integrals in (B.4.2) vanish.

Let $E_1', E_2', \cdots, E_k'$ and $\omega_i \in \Omega_{E_i'}(E)$ be as in Corollary B.3.3. Assume that $E' = E_1'$ and $\omega = \omega_1$. Corollary B.3.3 implies that the integral in (B.4.2) does not vanish identically for some (E_i', ω_i). Then $i \neq 1$ and

$$\mathrm{Hom}_{\mathrm{U}(E)}(\omega_i\widehat{\otimes}\pi, \mathbb{C}) \neq 0.$$

Now theta dichotomy implies that

$$\mathrm{Hom}_{\mathrm{U}(E)}(\omega\widehat{\otimes}\pi, \mathbb{C}) = 0.$$

This finishes the proof. □

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