Higher dimensional word problems with applications to equational logic

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The word problem on a monoid admits two natural generalizations:

- The first one is the extension from monoids to categories. In this case the words become "paths" in a graph, and the equality of paths is a problem of commutation of diagrams.
- The second one is the extension from monoids to universal algebras. In this case, the words become "terms", and the word problem becomes derivation in equational logic from given equations.

Is it possible to unify these two generalizations?

In this paper, we answer as follows: the latter problem is nothing but a 2-dimensional word problem in a "2-monoid", which leads to the syntactical study of a 3-category. This crucial observation leads to the general problem for n-paths in an n-category, or even in an ∞-category. A lot of computations made by category theorists are 1-, 2- or 3-dimensional, and in fact, n-dimensional computations take place in an $n{+}1$-category. Furthermore, beyond the unity thus given to various Thue problems, the link with combinatorial topology appears, rewriting systems being in this setting a refinement of homotopy theory.

Some ideas of this paper, which is an extended version of [Bu 3] have their origin in the dimensional analysis of formal languages [Bu 1] and the "elimination" of the universal property of cartesian product [Bu 2]. Theorem 1 was first communicated in March 1986 (séminaire Bénabou, I.H.P., Paris), theorem 2 in June 1988 (journées E.L.I.T., Paris) and section 1 in July 1989 (International Category Theory Meeting, Bangor).

1 General setting

We are going to define the n-dimensional word problem ($n \in \mathbf{N}$), and more generally, a word problem of variable dimension which means computing in an ∞-category (such a computation being often a reduction to a canonical form, but more generally being the construction of a "homotopy" between two expressions).

1.1 ∞-categories

The definition presented below is not the most synthetic one possible, but it fits our purpose pretty well. Finally the whole construction amounts to the juxtaposition of an infinity of 2-categories (see [Bou], page 411). An ∞-*graph* G is given by a diagram of sets

$$G_0 \overset{a_0}{\underset{b_0}{\leftleftarrows}} G_1 \overset{a_1}{\underset{b_1}{\leftleftarrows}} \cdots \overset{a_{n-1}}{\underset{b_{n-1}}{\leftleftarrows}} G_n \overset{a_n}{\underset{b_n}{\leftleftarrows}} G_{n+1} \overset{a_{n+1}}{\underset{b_{n+1}}{\leftleftarrows}} \cdots$$

such that, for every $n \in \mathbf{N}$, the following equations hold:

$$a_n a_{n+1} = a_n b_{n+1}, \qquad b_n a_{n+1} = b_n b_{n+1}.$$

The elements of G_n are named n-*cells*, and the following representation of 0-, 1- and 2-cells are well-known:

$$X \quad (n=0) \qquad\qquad X \longrightarrow Y \quad (n=1) \qquad\qquad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow\lambda"] & Y \end{tikzcd} \quad (n=2)$$

where $X, Y \in G_0$, $f, g \in G_1$ and $\lambda \in G_2$. Symbols are sometimes omitted, as for instance here:

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\downarrow\lambda"] & \cdot \end{tikzcd}$$

We also need higher dimensional cells, and representing them is possible (although difficult) as, for instance, for a 3-cell:

$$\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\downarrow\!\rightarrow\!\downarrow"] & \cdot \end{tikzcd}$$

An n-*graph* is defined in a similar way, by a finite sequence

$$G_0 \overset{a_0}{\underset{b_0}{\leftleftarrows}} G_1 \overset{a_1}{\underset{b_1}{\leftleftarrows}} \cdots \overset{a_{n-1}}{\underset{b_{n-1}}{\leftleftarrows}} G_n.$$

In particular, a 0-graph is just a set, and a 1-graph is an (oriented) graph in the usual sense. But the crucial notion here is the one of 2-graph (and 2-category).

Starting from an ∞-graph C we define, for $0 \le i < j$, a graph C_{ij}:

$$C_i \overset{a_{ij}}{\underset{b_{ij}}{\leftleftarrows}} C_j$$

with $a_{ij} = a_i a_{i+1} \cdots a_{j-1}$, $b_{ij} = b_i b_{i+1} \cdots b_{j-1}$, and we define, for $0 \le i < j < k$, a 2-graph C_{ijk}:

$$C_i \overset{a_{ij}}{\underset{b_{ij}}{\leftleftarrows}} C_j \overset{a_{jk}}{\underset{b_{jk}}{\leftleftarrows}} C_k$$

In order to get a structure of ∞-*category* on an ∞-graph C, we need only a category structure on each C_{ij} in such a way that the 2-graphs C_{ijk} become 2-categories. Essentially such a *2-category* is a 2-graph

$$C_0 \overset{a_0}{\underset{b_0}{\leftleftarrows}} C_1 \overset{a_1}{\underset{b_1}{\leftleftarrows}} C_2$$

with the following additional data:

i) a category structure on the graph $C_0 \overset{a_0}{\underset{b_0}{\leftleftarrows}} C_1$:

$$X \quad \mapsto \quad X \xrightarrow{\mathrm{id_X}} X, \qquad X \xrightarrow{f} Y \xrightarrow{g} Z \quad \mapsto \quad X \xrightarrow{gf} Z,$$

with the usual associativity and unit conditions,

ii) a category structure on the graph $C_0 \overset{a_0 a_1}{\underset{b_0 b_1}{\leftleftarrows}} C_2$, extending in some sense the previous one:

$$X \mapsto X \overset{\mathrm{id_X}}{\underset{\mathrm{id_X}}{\Downarrow \mathrm{Id_X}}} X, \qquad X \overset{f}{\underset{f'}{\Downarrow\lambda}} Y \overset{g}{\underset{g'}{\Downarrow\mu}} Z \mapsto X \overset{gf}{\underset{g'f'}{\Downarrow\mu\lambda}} Z,$$

iii) a category structure on the graph $C_1 \overset{a_1}{\underset{b_1}{\leftleftarrows}} C_2$, orthogonal in some sense to the previous one:

$$X \xrightarrow{f} Y \mapsto X \overset{f}{\underset{f}{\Downarrow \mathrm{id(f)}}} Y, \qquad X \overset{f}{\underset{f''}{\Downarrow\lambda \atop \Downarrow\lambda'}} Y \mapsto X \overset{f}{\underset{f''}{\Downarrow\lambda' \circ \lambda}} Y,$$

iv) finally we add compatibility conditions between structures (ii) and (iii):

$$\mathrm{Id_X} = \mathrm{id(id_X)}, \qquad (\mu' \circ \mu)(\lambda' \circ \lambda) = \mu'\lambda' \circ \mu\lambda$$

for all $X \in C_0$ and for all "2-diagrams" of the form $X \underset{\Downarrow\lambda'}{\overset{\Downarrow\lambda}{\to}} Y \underset{\Downarrow\mu'}{\overset{\Downarrow\mu}{\to}} Z$.

In practise, a consequence of the latter equation is used (Godement's rule):

$$\mathrm{id(g')}\lambda \circ \mu\,\mathrm{id(f)} = \mu\,\mathrm{id(f')} \circ \mathrm{id(g)}\lambda$$

for all diagrams $X \overset{f}{\underset{f'}{\Downarrow\lambda}} Y \overset{g}{\underset{g'}{\Downarrow\mu}} Z$.

An *n-category* is defined in a similar way on an n-graph. Now we have the usual notions of homomorphism between ∞-graphs or ∞-categories, which we define as natural transformations through diagrams (with conditions of commutation and projective limits). So we get two categories, namely ∞-**Cat** and ∞-**Graph**, and a forgetful functor

$$\mathbf{U}_\infty : \infty\text{-}\mathbf{Cat} \to \infty\text{-}\mathbf{Graph},$$

and from general category theory, this functor $\mathbf{U}_\infty$ admits a left adjoint $\mathbf{F}_\infty$. The ∞-category $\mathbf{F}_\infty(\mathbf{G})$, which we shall also write $[G]$ is named the *free ∞-category* generated by the ∞-graph G. We also know through general arguments that ∞-**Cat** and ∞-**Graph** are complete and co-complete. In the same way we get a functor

$$\mathbf{U_n} : \mathbf{n\text{-}Cat} \to \mathbf{n\text{-}Graph}$$

with a left adjoint $\mathbf{F_n}$, and we have a double commutative square

$$\begin{tikzcd} n\text{-}\mathbf{Cat} \arrow[r, "\mathbf{J_n}"] \arrow[d, shift right, "\mathbf{U_n}"'] & \infty\text{-}\mathbf{Cat} \arrow[d, shift right, "\mathbf{U}_\infty"'] \\ n\text{-}\mathbf{Graph} \arrow[r, "\mathbf{I_n}"] \arrow[u, shift right, "\mathbf{F_n}"'] & \infty\text{-}\mathbf{Graph} \arrow[u, shift right, "\mathbf{F}_\infty"'] \end{tikzcd}$$

where $\mathbf{I_n}$ is the functor which extends an n-graph by $G_p = \emptyset$ for $p > n$ and $\mathbf{J_n}$ is the functor that adds units in higher dimensions. This is why we can embed all constructions on n-**Cat** and n-**Graph** into the single category ∞-**Cat**.

1.2 CW-presentations

The ∞-categories can be constructed by adjoining successive cells as for the CW-complexes, these being completed by constructions of collapsing. This adjoining/collapsing pair is the analog of the generator/relation pair in the description of algebraic structures. The *formal n-cell*, for $n \in \mathbf{N}$, is the free ∞-category $[\mathbf{e_n}]$ generated by the ∞-graph $\mathbf{e_n}$ which is defined by

$$(\mathbf{e_n})_\mathbf{p} = \begin{cases} 2 & \text{if } 0 \leq p < n, \\ 1 & \text{if } p = n, \\ 0 & \text{if } p > n, \end{cases}$$

with $a_p, b_p : 2 \to 2$ for $0 \leq p < n{-}1$, and $a_{n-1}, b_{n-1} : 1 \to 2$, given as the constant map on 0 and 1 respectively (we adopt Von Neuman's convention, namely $p = \{0, 1, \ldots, p{-}1\}$). In particular,

$$\mathbf{e_0} \quad = \quad (\cdot), \qquad \mathbf{e_1} \quad = \quad (\cdot \longrightarrow \cdot), \qquad \mathbf{e_2} \quad = \quad \left(\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\downarrow"] & \cdot \end{tikzcd}\right).$$

We define the *boundary* $\partial\mathbf{e_n}$ as being $\mathbf{e_n}$ truncated at dimension $n{-}1$:

$$(\partial\mathbf{e_n})_\mathbf{p} = \begin{cases} (\mathbf{e_n})_\mathbf{p} & \text{if } 0 \leq p < n, \\ 0 & \text{if } p \geq n, \end{cases}$$

For example,

$$\partial\mathbf{e_0} \quad = \quad (\,), \qquad \partial\mathbf{e_1} \quad = \quad (\cdot \qquad \cdot), \qquad \partial\mathbf{e_2} \quad = \quad \left(\begin{tikzcd} \cdot \arrow[r, bend left] \arrow[r, bend right] & \cdot \end{tikzcd}\right).$$

We also define, for all $n \geq 1$, three obvious ∞-functors making a commutative diagram

$$\begin{tikzcd}
{[\mathbf{e_{n-1}}]} & & {[\mathbf{e_n}]} \arrow[ll, "\mathbf{triv_n}"'] \\
& {[\partial\mathbf{e_n}]} \arrow[ul, "\mathbf{col_n}"] \arrow[ur, "\mathbf{adj_n}"'] &
\end{tikzcd}$$

and for $n = 0$, we have only $\mathbf{adj_0} : [\partial\mathbf{e_0}] \to [\mathbf{e_0}]$. Note that $\mathbf{adj_n}$ is a monomorphism, whereas $\mathbf{col_n}$ and $\mathbf{triv_n}$ are surjective epimorphisms.

An $(n{-}1)$-dimensional *attaching map* to an ∞-category C is an ∞-functor $\varphi : [\partial\mathbf{e_n}] \to \mathbf{C}$. Given such a φ, we associate two ∞-categories C/φ and $C[\varphi]$, by means of two pushouts, namely the internal squares of the following commutative diagram:

$$\begin{tikzcd}
{[\mathbf{e_{n-1}}]} \arrow[dddd] & & {[\mathbf{e_n}]} \arrow[ll, "\mathbf{triv_n}"'] \arrow[dddd] \\
& {[\partial\mathbf{e_n}]} \arrow[ul, "\mathbf{col_n}"] \arrow[ur, "\mathbf{adj_n}"'] \arrow[dd, "\varphi"] & \\
& & \\
& C \arrow[dl] \arrow[dr] & \\
C/\varphi & & C[\varphi] \arrow[ll]
\end{tikzcd}$$

C/φ is called the *collapsing* (of dimension $n-1$) of C by φ, and $C[\varphi]$ is called the *adjoining* (of dimension n) of φ to C. The relationship between adjoining and collapsing is very strong. For example if C is a $(n-1)$-category and $\varphi : [\partial \mathbf{e_n}] \to \mathbf{C}$ is an attaching map, then $C[\varphi]$ is a sort of syntax for C/φ, because the study of the latter takes place in $C[\varphi]$, via the surjective epimorphism $C[\varphi] \to C/\varphi$.

We say that an ∞-category C is *finitely presented* if there exists a finite sequence of ∞-categories

$$\emptyset = C_0,\ C_1,\ \ldots,\ C_{m-1},\ C_m = C$$

such that, for all $0 \leq i < m$, $C_{i+1} = C_i/\varphi$ or $C_{i+1} = C_i[\varphi]$, where φ is an attaching map to C_i.

Of course, this is just one possible way of presenting an ∞-category. The ability of interleaving adjoinings and collapsings may be crucial in the general case, and this flexibility is also useful in studying algebraic theories (see section 2). However, in the sequel, we shall only consider the case of adjoining cells with non decreasing dimension, and finally collapsing cells of maximal dimension $n = \dim C$. Then, if $(\varphi_i)_{1 \leq i \leq p}$ is the sequence of adjoinings and $(\psi_i)_{1 \leq i \leq q}$ is the sequence of collapsings, we write

$$C = \emptyset[\varphi_1, \ldots, \varphi_p]/(\psi_1, \ldots, \psi_q),$$

and we call this a *CW-presentation* of C, in order to emphasize the analogy with CW-complexes in topology.

Before presenting a structure implementing such a construction (in a slightly different form, where cells are added by blocks dimensionwise), let us explain informally why we accept that all collapsings be of dimension n. First, it is clear that they cannot be of higher dimension. Next, if we limit ourselves to the case where $n = 1$, which is so familiar to category theorists, adding an invertible morphism (*i.e.* adjoining two 1-cells $f : X \to Y$, $g : Y \to X$ and then collapsing in dimension 1 by the relations $gf = \mathrm{id_X}$, $fg = \mathrm{id_Y}$) is preferred to equalizing objects (collapsing by $X = Y$ in dimension 0). The result is of course equivalent in the categorical sense.

1.3 Polygraphs

The structure of n-graph ($n \in \mathbf{N}$ or $n = \infty$) defined in 1.1 is too rudimentary to be really useful in the study of n-categories when $n \geq 2$. In particular, it is well known that it is inadequate for a theoretical definition of the notion of "2-diagram", *i.e.* a diagram with 2-cells, as the following one:

$$\begin{tikzcd} & \cdot \arrow[dr] \arrow[d, phantom, "\downarrow"] & \\ \cdot \arrow[ur] \arrow[rr] & {} & \cdot \end{tikzcd}$$

Indeed, in this kind of diagram, the source and the target of a 2-cell are "paths" and not necessarily single 1-cells. The traditional commutative diagrams are examples of 2-diagrams if we consider commuting equations as 2-cells.

The notion that seems to emerge from this kind of example is the structure of an n-polygraph, which is more general than the structure of an n-graph (they coincide for $n \leq 1$). Before giving the precise definition, consider the category n-$\mathbf{Cat}^+$

defined by the pullback of categories

$$
\begin{tikzcd}
n\text{-}\mathbf{Cat}^+ \arrow[r, "\mathbf{U'_n}"] \arrow[d, "\mathbf{V'_n}"'] & n{+}1\text{-}\mathbf{Graph} \arrow[d, "\mathbf{V_n}"] \\
n\text{-}\mathbf{Cat} \arrow[r, "\mathbf{U_n}"'] & n\text{-}\mathbf{Graph}
\end{tikzcd}
$$

where $\mathbf{U_n}, \mathbf{V_n}$ are the forgetful functors and $\mathbf{U'_n}, \mathbf{V'_n}$ are the canonical projections of the pullback. Clearly an object of n-$\mathbf{Cat}^+$ is determined by an n-category C on an n-graph

$$C_0 \leftleftarrows C_1 \leftleftarrows \cdots \leftleftarrows C_n$$

together with a set G_{n+1} and two maps $G_{n+1} \to C_n$ such that

$$C_0 \leftleftarrows C_1 \leftleftarrows \cdots \leftleftarrows C_n \leftleftarrows G_{n+1}$$

is an n+1-graph. Now we have a new forgetful functor $\mathbf{W_n} : \mathbf{n{+}1\text{-}Cat} \to \mathbf{n\text{-}Cat}^+$ which forgets the compositions on n+1-cells but keeps the cells.

Lemma 1 *The functor $\mathbf{W_n}$ has a left adjoint $\mathbf{L_n}$ such that $\mathbf{V'_n W_n L_n}$ is the forgetful functor from n+1-$\mathbf{Cat}$ to n-$\mathbf{Cat}$ (i.e. $\mathbf{W_n}$ preserves the p-cells and their composition for $p \leq n$).*

The proof is a matter of general category theory. In fact, the construction is essentially the one of adjoining cells, except that the set of adjoined cells is not supposed to be finite.

Now we give the main definition. An *n-polygraph* consists of a diagram

$$
\begin{tikzcd}
\Sigma_0 \arrow[d, "i_0"'] & \Sigma_1 \arrow[d, "i_1"'] \arrow[dl, shift right, "a_0"'] \arrow[dl, shift left, "b_0"] & \Sigma_2 \arrow[d, "i_2"'] \arrow[dl, shift right, "a_1"'] \arrow[dl, shift left, "b_1"] & \cdots & \Sigma_{n-2} \arrow[d, "i_{n-2}"'] & \Sigma_{n-1} \arrow[d, "i_{n-1}"'] \arrow[dl, shift right, "a_{n-2}"'] \arrow[dl, shift left, "b_{n-2}"] & \Sigma_n \arrow[dl, shift right, "a_{n-1}"'] \arrow[dl, shift left, "b_{n-1}"] \\
\Sigma_0^* & \Sigma_1^* \arrow[l, shift right, "\overline{a}_0"'] \arrow[l, shift left, "\overline{b}_0"] & \Sigma_2^* \arrow[l, shift right, "\overline{a}_1"'] \arrow[l, shift left, "\overline{b}_1"] & \cdots & \Sigma_{n-2}^* & \Sigma_{n-1}^* \arrow[l, shift right, "\overline{a}_{n-2}"'] \arrow[l, shift left, "\overline{b}_{n-2}"] &
\end{tikzcd}
$$

together with, for each p $(0 \leq p < n)$, a structure of p-category $\overline{\Sigma}_p$ on the graph

$$\Sigma_0^* \underset{\overline{b}_0}{\overset{\overline{a}_0}{\leftleftarrows}} \Sigma_1^* \underset{\overline{b}_1}{\overset{\overline{a}_1}{\leftleftarrows}} \cdots \underset{\overline{b}_{p-1}}{\overset{\overline{a}_{p-1}}{\leftleftarrows}} \Sigma_p^*$$

and such that

$$\Sigma_0^* \underset{\overline{b}_0}{\overset{\overline{a}_0}{\leftleftarrows}} \Sigma_1^* \underset{\overline{b}_1}{\overset{\overline{a}_1}{\leftleftarrows}} \cdots \underset{\overline{b}_{p-1}}{\overset{\overline{a}_{p-1}}{\leftleftarrows}} \Sigma_p^* \underset{b_p}{\overset{a_p}{\leftleftarrows}} \Sigma_{p+1}$$

is a p+1-graph. Finally, writing Σ_p^+ for the corresponding object in p-$\mathbf{Cat}^+$, we require that, if $p+1 < n$, then $\overline{\Sigma}_{p+1} = \mathbf{L_p}(\mathbf{\Sigma_p^+})$ and $i_{p+1} : \Sigma_{p+1} \to \Sigma_{p+1}^*$ is the corresponding universal map (in particular $\overline{a}_p i_{p+1} = a_p$ and $\overline{b}_p i_{p+1} = b_p$). The last part of the data

$$\Sigma_{n-1} \underset{b_{n-1}}{\overset{a_{n-1}}{\leftleftarrows}} \Sigma_n$$

can be considered as a description of collapsings, in accordance with what was suggested in 1.2.

Now an *∞-polygraph*, or simply a *polygraph*, is defined similarly, without limitations on the right. Clearly, any n-polygraph can be considered as a polygraph by extending data trivially on the right.

1.4 Examples

The complexity of the above construction increases with the dimension n.

1.4.1 Case $n = 0$

The 0-dimensional word problem can be identified with the so-called "maze problem". Let $\Sigma_0 \leftleftarrows \Sigma_1$ be a graph and $X, Y \in \Sigma_0$. Does there exist a path from X to Y? To this semi-Thue problem corresponds a Thue problem, which is the actual maze problem: is X identified with Y in the quotient $\Sigma_0/(\Sigma_1)$ of Σ_0 by the equivalence relation generated by the graph? Then, in the diagram of 1.3 for $n = 0$, Σ_0 represents a 0-graph and Σ_0^* is the 0-category generated by Σ_0. Of course, since there are zero operations (in a n-category there are n operations on the n-cells), these two structures can be identified with the set $\Sigma_0 = \Sigma_0^*$ and i_0 is the identity. One can also consider Σ_0 as a set of 0-cells adjoined to $\emptyset$. The graph $\Sigma_0 = \Sigma_0^* \leftleftarrows \Sigma_1$ represents the adjoining of 1-cells which are used to describe the collapsing $\Sigma_0/(\Sigma_1)$.

1.4.2 Case $n = 1$

We obtain the classical word problem in monoids (if $\Sigma_0 = 1$, in which case $a_0, b_0 : \Sigma_1 \to 1$ are too trivial to appear) and more generally in categories. Σ_1 is what is usually called the *alphabet*. The data $\Sigma_0 = \Sigma_0^* \leftleftarrows \Sigma_1^*$ represent the underlying graph of the free category generated by the graph $\Sigma_0^* \leftleftarrows \Sigma_1$, and $i_1 : \Sigma_1 \to \Sigma_1^*$ is the canonical embedding at level 1 of a graph Σ into the underlying graph of the free category Σ^* generated by it. $\Sigma_1^* \leftleftarrows \Sigma_2$ represents a set of diagrams of the form

$$\cdot \quad \begin{array}{c} \xrightarrow{f_1} \cdot \xrightarrow{f_2} \cdots \xrightarrow{f_{p-1}} \cdot \xrightarrow{f_p} \\ \downarrow\lambda \\ \xrightarrow[g_1]{} \cdot \xrightarrow[g_2]{} \cdots \xrightarrow[g_{q-1}]{} \cdot \xrightarrow[g_q]{} \end{array} \quad \cdot$$

i.e. the adjoining of a set of 2-cells $\lambda \in \Sigma_2$. As before, this problem entails a semi-Thue problem and the associated Thue problem is the classical word problem. The elements of Σ_2^* represent "2-paths" admitting a unique decomposition, which is schematically given in figure 1. In this figure, 1-cells represent in fact paths, *i.e.* elements of Σ_1^*, 2-cells represent elements of Σ_2, and the "=" means commutativity in the free category generated by the graph $\Sigma_0^* \leftleftarrows \Sigma_1$.

1.4.3 Case $n = 2$

The construction goes on with the adjoining of 3-cells $\Sigma_2^* \leftleftarrows \Sigma_3$. It is easy to imagine, but tedious to describe, how the elements of Σ_3^* are constructed. Let us merely say that the corresponding Thue problem is nothing but a computation in 2-categories, or the "2-calculus". It has recently become an object of research, especially by P.L. Curien [Cur] and Y. Lafont [Laf], after the works of E.G. Rodeja and his school in Santiago de Compostella. Finally it is also related to the rewriting problem for terms as we shall see in part 2.

1.5 Relationship with combinatorial topology

Considered as geometrical objects, the ∞-graphs are too poor to give interesting models. On the other hand, simplicial sets, cubic sets, and possibly other objects of this kind, provide the classical models of combinatorial topology. But all of them (∞-graphs, simplicial sets, etc.) can be interpreted as polygraphs. It is obvious for

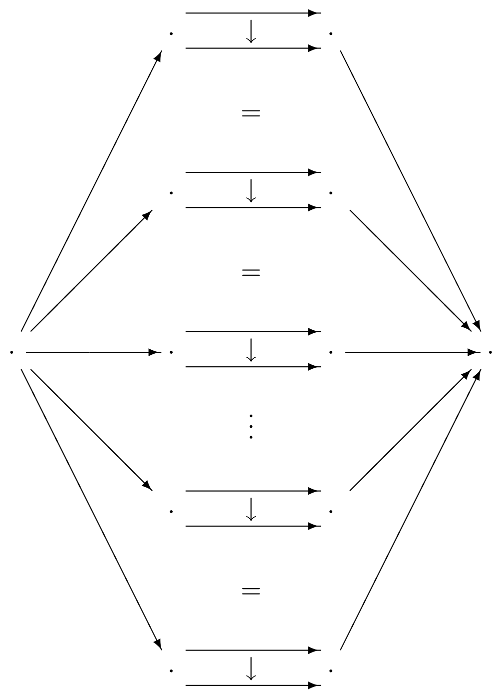


Figure 1: 2-path

∞-graphs, but even if the intuition strongly suggests that this is true for simplicial and cubic sets, an explicit realization of this fact seems to be a very complex problem. That has been achieved for simplicial sets in the work of R. Street [Str]. To illustrate this, we consider a cube and a hypercube, and freely interpret them as a polygraph. In figure 2 and 3, the various kinds of arrows represent p-cells for $p = 1, 2, 3, 4$. For example, the eight octagons in figure 3, which are "directed" surfaces consisting of six little squares, are linked via little cubes. In fact, all the boundaries of those octagons must be identified (as the hexagons of figure 2). They have been duplicated to make the picture readable.

But there is an essential difference between the word problems and the combinatorial problems: here all the cells are oriented and these orientations are essential, except in the collapsings where they have a purely technical function (Church-Rosser property).

2 Equational logic as a 2-dimensional word problem

The notion of an algebraic theory was defined by W. Lawvere [Law]. It is a strict cartesian category T in which every object is a cartesian power $\mathbf{s}^{\mathbf{n}} = \mathbf{s} \times \mathbf{s} \times \cdots \times \mathbf{s}$ ($n \in \mathbf{N}$) of a unique object $\mathbf{s}$ of T. In this second part we show how an algebraic theory can be thought of as a 2-category (in fact a 2-monoid) and how a finite equational system in "universal algebra" can be interpreted as a finite CW-presentation. So we are in the context of 2-dimensional word problems.

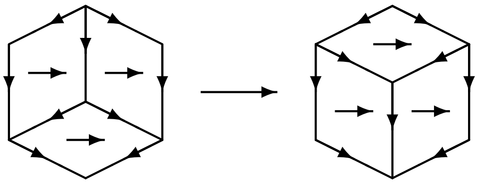


Figure 2: the cube as a 3-dimensional cell

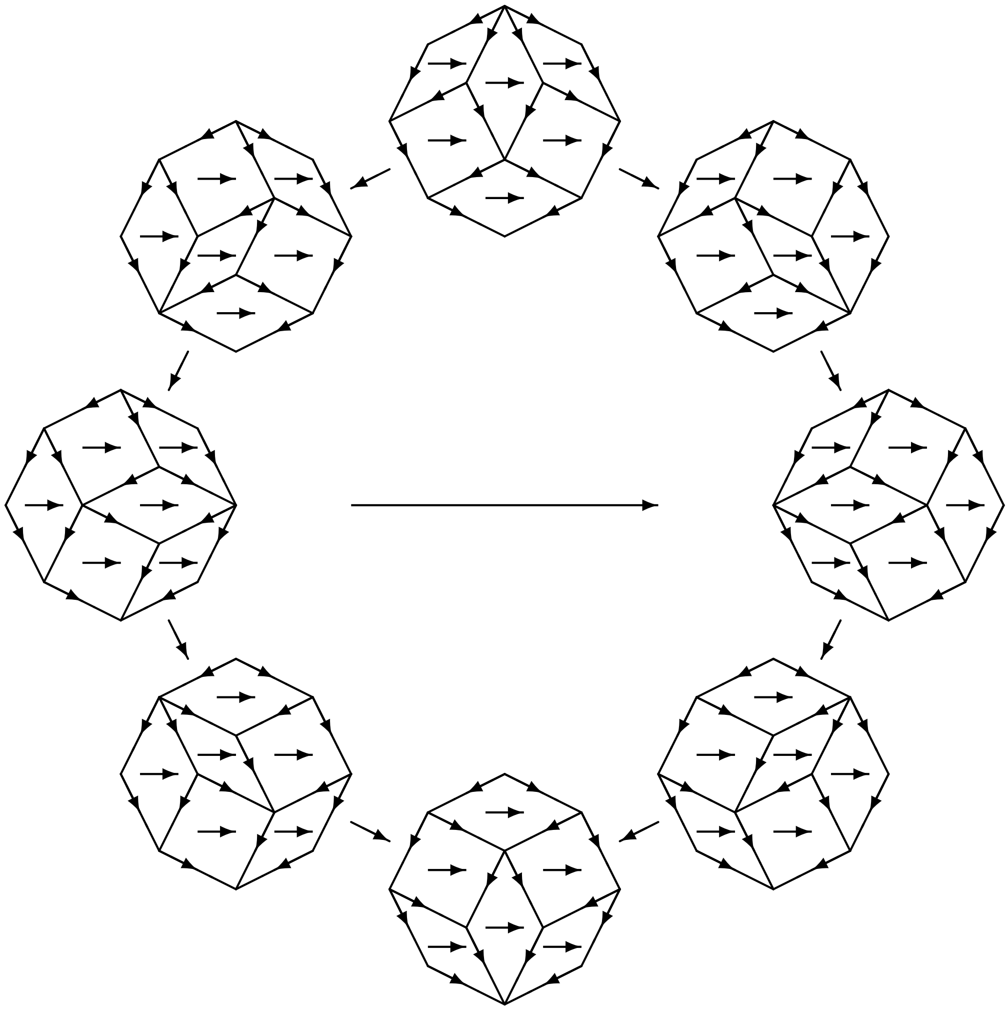


Figure 3: the hypercube as a 4-dimensional cell

2.1 2-monoids

Let $n \in \mathbf{N}$, and let us call *n-monoid* an n-category C such that $C_0 = \{*\}$ (*i.e.* a single 0-cell). A 1-monoid is of course a monoid. A 2-monoid can also be interpreted as a *strict monoidal category*, that is a 3-uple $(C, I, \otimes)$ where C is a category, I an object of C and $\otimes : C \times C \to C$ a functor satisfying the unit and associativity conditions:

$$\mathrm{id(I)} \otimes \mathrm{f} = \mathrm{f} = \mathrm{f} \otimes \mathrm{id(I)}, \qquad (\mathrm{f} \otimes \mathrm{g}) \otimes \mathrm{h} = \mathrm{f} \otimes (\mathrm{g} \otimes \mathrm{h})$$

for all morphisms f, g, h of C (the non strict monoidal categories are those where equalities are replaced with coherent natural equivalences). The transformation which exhibits $(C, I, \otimes)$ as a 2-monoid is similar to suspension in topology: the dimension of the elements of C is "augmented". First a 0-cell $*$ is added, then each object X of C becomes a 1-cell $X : * \longrightarrow *$ and finally, each morphism $f : X \longrightarrow Y$ of C becomes a 2-cell of the 2-monoid:

$$\begin{tikzcd} * \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\downarrow f"] & * \end{tikzcd}$$

A homomorphism $F : (C, I, \otimes) \to (C', I', \otimes')$ between two strict monoidal categories (considered as 2-categories) is a functor $F : C \to C'$ which commutes with the operations (strict monoidal functor). We write 2-**Mon** for the category of 2-monoids and homomorphisms between them. It is of course a full subcategory of 2-**Cat**.

An example of a monoidal category is given by a cartesian category $(C, 1, \times)$ where 1 is a terminal object of C and $\times : C \times C \to C$ is a cartesian product functor. Here we shall consider such categories as if they were always strict. In fact we do nothing but following the customs of algebraists and logicians (not distinguishing between $(\mathbf{s} \times \mathbf{s}) \times \mathbf{s}$, $\mathbf{s} \times (\mathbf{s} \times \mathbf{s})$ and $\mathbf{s} \times \mathbf{s} \times \mathbf{s}$). Furthermore the only cartesian category that will really interest us (see 2.2) is strict cartesian. A first step towards the "elimination" of the universal property of the cartesian product (for the benefit of an equational system) is given with the following result (see [Bu 1, Bu 2]):

Proposition 1 *The strict monoidal category $(C, 1, \times)$ is cartesian if and only if there are two natural transformations ε, δ which for every object X in C define two morphisms*

$$1 \xleftarrow{\varepsilon(X)} X \xrightarrow{\delta(X)} X \times X$$

satisfying the relations

$$(\mathrm{id(X)} \times \varepsilon(\mathrm{X})) \circ \delta(\mathrm{X}) = \mathrm{id(X)} = (\varepsilon(\mathrm{X}) \times \mathrm{id(X)}) \circ \delta(\mathrm{X}),$$

$$((\mathrm{id(X)} \times \varepsilon(\mathrm{Y})) \times (\varepsilon(\mathrm{X}) \times \mathrm{id(Y)})) \circ \delta(\mathrm{X} \times \mathrm{Y}) = \mathrm{id(X \times Y)}, \qquad \varepsilon(1) = \mathrm{id}(1),$$

for all objects X and Y in C. When they exist, these data are unique.

Unfortunately, even in the case of a finite equational system, there are infinitely many objects $\mathbf{s^0}, \mathbf{s^1}, \mathbf{s^2}, \ldots$ So we get infinitely many data $\varepsilon(\mathbf{s^n})$, $\delta(\mathbf{s^n})$ and infinitely many equations, including those expressing the naturality of ε and δ.

Now we shall dedicate the rest of this text to reducing such systems to finite presentations. In particular $\varepsilon(\mathbf{s^n})$ and $\delta(\mathbf{s^n})$ will be constructed using only $\varepsilon(\mathbf{s})$, $\delta(\mathbf{s})$ and $\sigma(\mathbf{s}, \mathbf{s})$, where $\sigma(\mathbf{s^p}, \mathbf{s^q}) : \mathbf{s^{p+q}} \to \mathbf{s^{p+q}}$ is given by the symmetry of the cartesian product. The latter is needed in the inductive building of $\delta(\mathbf{s^n})$ by means of the relation:

$$\delta(\mathbf{s^{p+1}}) = (\mathbf{s^p} \times \sigma(\mathbf{s^p}, \mathbf{1}) \times \mathbf{s}) \circ (\delta(\mathbf{s^p}) \times \delta(\mathbf{s})).$$

In 2.2, we shall do this for a special theory, namely the initial theory $\mathbf{F^{op}}$. In the dual category $\mathbf{F}$, $\varepsilon(\mathbf{s})$, $\delta(\mathbf{s})$ and $\sigma(\mathbf{s}, \mathbf{s})$ will be written respectively η, μ and τ.

2.2 A CW-presentation of F

Let **Set** be the category of sets and $\mathbf{F}$ be the full subcategory of **Set** whose objects are the natural numbers considered as finite sets ($n = \{0, 1, \ldots, n{-}1\}$). $\mathbf{F}$ and its dual $\mathbf{F^{op}}$ are both strict cartesian categories, but here we shall only use $\mathbf{F^{op}}$, or if you prefer, the 2-monoid $(\mathbf{F}, \mathbf{0}, +)$ where 0 is the empty set and + the coproduct functor in $\mathbf{F}$. The latter coincides with addition on natural numbers. Precisely, for each $u : p \to q$ and $u' : p' \to q'$, the map $u + u' : p + p' \to q + q'$ is defined by

$$(u + u')(i) = \begin{cases} u(i) & \text{if } 0 \leq i < p, \\ u'(i - p) & \text{if } p \leq i < p + p'. \end{cases}$$

Theorem 1 *The 2-monoid* $(\mathbf{F}, \mathbf{0}, +)$ *is finitely presented (in the sense of CW-presentations of 1.2).*

Before giving the proof, we will first give the explicit CW-presentation of this 2-monoid. The adjoinings are:

- one 0-cell $\quad *\quad$,
- one 1-cell $\quad * \xrightarrow{1} * \quad$,
- three 2-cells $\begin{tikzcd} * \arrow[r, shift left=2, "0"] \arrow[r, shift right=2, "1"'] \arrow[r, phantom, "\Downarrow\eta"] & * \end{tikzcd}$, $\begin{tikzcd} * \arrow[r, shift left=2, "2"] \arrow[r, shift right=2, "1"'] \arrow[r, phantom, "\Downarrow\mu"] & * \end{tikzcd}$ and $\begin{tikzcd} * \arrow[r, shift left=2, "2"] \arrow[r, shift right=2, "2"'] \arrow[r, phantom, "\Downarrow\tau"] & * \end{tikzcd}$ where 2 represents 1+1.

The collapsings are written hereafter by means of equations instead of arrows, which leaves us free to choose their orientations (for a choice giving a canonical system, see [Laf]). We do not try to avoid redundancy (and we write n instead of id(n)):

$$(1)\quad \mu \circ (\eta + 1) = 1, \qquad (2)\quad \mu \circ (1 + \eta) = 1, \qquad (3)\quad \mu \circ (\mu + 1) = \mu \circ (1 + \mu),$$

$$(4)\quad \tau \circ (1 + \eta) = \eta + 1, \qquad (5)\quad \tau \circ (\eta + 1) = 1 + \eta,$$

$$(6)\quad \tau \circ (\mu + 1) = (1 + \mu) \circ (\tau + 1) \circ (1 + \tau),$$

$$(7)\quad \tau \circ (1 + \mu) = (\mu + 1) \circ (1 + \tau) \circ (\tau + 1),$$

$$(8)\quad \tau \circ \tau = 2, \qquad (9)\quad (1 + \tau) \circ (\tau + 1) \circ (1 + \tau) = (\tau + 1) \circ (1 + \tau) \circ (\tau + 1),$$

$$(10)\quad \mu \circ \tau = \mu.$$

In an earlier formulation of this result, the relation (10), which plays a subtle role in this system, was missing. We are thankful to Y. Lafont for pointing this out to us. A graphical schematization of these equations is given in figure 4.

$\mathbf{F}$ can also be presented by the polygraph

$$\begin{tikzcd} \{*\} \arrow[d] & \{1\} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[d] & \{\eta, \mu, \tau\} \arrow[dl, shift left] \arrow[dl, shift right] \arrow[d] & \Theta \arrow[dl, shift left] \arrow[dl, shift right] \\ \{*\}^* & \{1\}^* \arrow[l, shift left] \arrow[l, shift right] & \{\eta, \mu, \tau\}^* \arrow[l, shift left] \arrow[l, shift right] & \end{tikzcd}$$

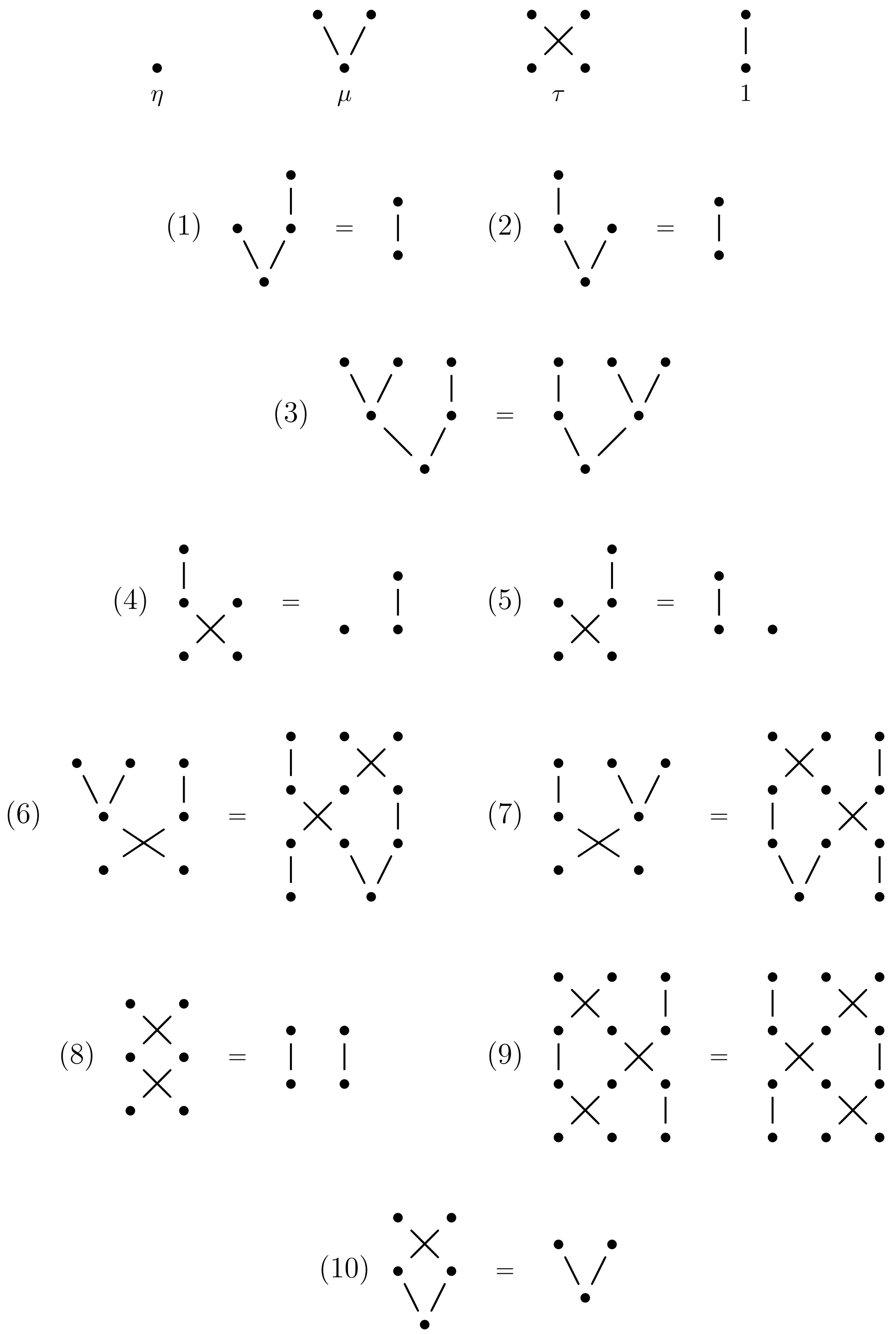


Figure 4: presentation of **F**

where Θ is the collapsing system given above (equations (1) to 10)). We shall write $\widetilde{\mathbf{F}}$ for the 2-category underlying this polygraph (so forgetting Θ), and $=_\Theta$ for the congruence generated by Θ on the 2-cells of $\widetilde{\mathbf{F}}$. In particular, if $F =_\Theta F'$, then $a(F) = a(F')$ and $b(F) = b(F')$. Furthermore, we have a functor $\pi : \widetilde{\mathbf{F}} \to \mathbf{F}$ defined by $\pi(*) = *$, $\pi(1) = 1$ (we identify $\{1\}^*$ with $\mathbf{N}$) and $\pi(\eta), \pi(\mu), \pi(\tau)$ which are simply written η, μ, τ are the maps $\eta : 0 \to 1$, $\mu : 2 \to 1$, $\tau : 2 \to 2$ where the latter is defined by $\tau(0) = 1$ and $\tau(1) = 0$ (transposition).

Proof of Theorem 1: Every map $f : p \to q$ ($p, q \in \mathbf{N}$) has a unique decomposition

$$f = f_\eta \circ f_\mu \circ f_\tau$$

where the components are determined by the following conditions:

- f_η is a monotone injection,
- f_μ is a monotone surjection,
- f_τ is a f-*regular* bijection.

The third condition means that for all $j \in q$, the restriction of f_τ to the subset $f^{-1}(j)$ of p is monotone. The condition that f_τ be a bijection would clearly not be enough (a minimal counter-example is given by equation (10) written $\mu \circ \tau = \mu \circ 2$).

In particular, this decomposition shows that $\pi : \widetilde{\mathbf{F}} \to \mathbf{F}$ is surjective since the 2-monoid $\mathbf{F}$ is clearly generated by the maps η, μ, τ. Indeed, f_η can be decomposed into maps of the form $i + \tau + i'$, and similarly for f_μ and f_τ. The case of f_τ corresponds to the fact that permutations are generated by certain transpositions. Now, proving the theorem amounts to showing that $F =_\Theta F'$ if and only if $\pi(F) = \pi(F')$ for all $F, F' : p \to q$, 2-paths in $\widetilde{\mathbf{F}}$. Clearly $F =_\Theta F'$ entails $\pi(F) = \pi(F')$ (validity of the system Θ). We shall prove the converse.

We shall call η-path (respectively μ-path and τ-path) a 2-path $F : p \to q$ which uses only the 2-cell η (respectively μ and τ). The result is easily proved in the case of η-paths and μ-paths (using (3)). For τ-paths, using (8) and (9), it is possible to adapt a classical presentation of symmetric groups (see [Cox], page 63).

We shall introduce, for every map $f : p \to q$, a 2-path $\rho(f) : p \to q$ in $\widetilde{\mathbf{F}}$, called the *canonical form* of f, such that $\pi(\rho(f)) = f$ (but ρ will not be functorial). It is indeed easy to describe such a canonical form in the special cases where f is a monotone injection (respectively a monotone surjection and a bijection) using only the 2-cell η (respectively μ and τ). Several choices are possible, but we assume some has been made. If $f = f_\eta \circ f_\mu \circ f_\tau$, we define $\rho(f) = F$ by $F = F_\eta \circ F_\mu \circ F_\tau$ where $F_\eta = \rho(f_\eta)$, $F_\mu = \rho(f_\mu)$ and $F_\tau = \rho(f_\tau)$ have been fixed as suggested above. Obviously $\pi(F) = f$.

Now we must prove that for all $F' : p \to q$ in $\widetilde{\mathbf{F}}$ such that $\pi(F') = f$ we have $F' =_\Theta F$. By the equations (1), (2), (4), (5), (6) and (7) , there is $F'' : p \to q$ in $\widetilde{\mathbf{F}}$ such that $F'' =_\Theta F'$ and $F'' = K \circ H \circ G$ where K is an η-path, H a μ-path and G a τ-path. Clearly $\pi(F'') = f$, $\pi(K)$ is a monotone injection, $\pi(H)$ a monotone surjection and $\pi(G)$ a bijection. Therefore $K =_\Theta F_\eta$, $H =_\Theta F_\mu$ and $f = f_\eta \circ f_\mu \circ \pi(G)$, so $f_\eta \circ f_\mu = f_\eta \circ f_\mu \circ (\pi(G) \circ f_\tau^{-1})$. Since $f_\eta \circ f_\mu$ is monotone and $\pi(G) \circ f_\tau^{-1}$ is bijective, it is easy to see that

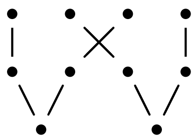
- $f_\eta \circ f_\mu = f_0 + f_1 + \cdots + f_{q-1}$ with $f_j : p_j \to 1$ for all $j \in q$, and $p_0 + p_1 + \cdots + p_{q-1} = p$,
- $\pi(G) \circ f_\tau^{-1} = g_0 + g_1 + \cdots + g_{q-1}$ where $g_j : p_j \to p_j$ is bijective for all $j \in q$.

Therefore, it is enough to consider the case where $q = 1$, which is solved by an iterated (and tedious) application of equations (3) and (10). **Q.e.d.**

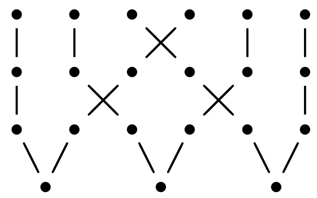
In 2.3, we shall use maps of the form $\eta_p : 0 \to p$, $\mu_p : p+p \to p$ and $\tau_{p,q} : p+q \to p+q$ (written in the dual form ε_p, δ_p and $\sigma_{p,q}$). We exhibit three examples of their decomposition in this presentation:

- $\mu_2 = (\mu + \mu) \circ (1 + \tau + 1)$,
- $\mu_3 = (\mu + \mu + \mu) \circ (1 + \tau + \tau + 1) \circ (2 + \tau + 2)$,
- $\tau_{2,3} = (1 + \tau + 2) \circ (\tau + \tau + 1) \circ (1 + \tau + \tau) \circ (2 + \tau + 1)$,

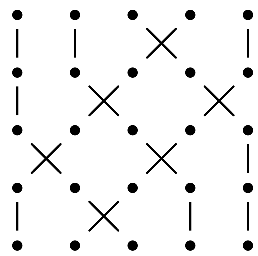
which can be pictured as follows:



μ_2



μ_3



$\tau_{2,3}$

2.3 The main theorem

An algebraic theory (with one sort) is presented by a system (Ω, E) of operations and equations. This generates a strict cartesian category $T = \mathbf{T}(\boldsymbol{\Omega}, \mathbf{E})$ characterized by the following universal property: an (Ω, E)-algebra in a strict cartesian category C amounts to a strict cartesian functor $F : T \to C$. In fact, if we take as canonical projections the products

$$X \xleftarrow{\ \mathrm{id(X)} \times \varepsilon\mathrm{(Y)}\ } X \times Y \xrightarrow{\ \varepsilon(X) \times \mathrm{id(Y)}\ } Y,$$

it suffices to assume that F is a strict monoidal functor, because of the relations $F(1) = 1$, $F(f \times g) = F(f) \times F(g)$ and $F(\varepsilon(X)) = \varepsilon(F(X)) : F(X) \to 1$ (since 1 is terminal in C).

If $\Omega = \emptyset$ and $E = \emptyset$, the corresponding theory is nothing but $\mathbf{T}(\emptyset, \emptyset) = \mathbf{F^{op}}$ ("theory of sets"). More precisely, the following universal property of $\mathbf{F^{op}}$ is well known: for every strict cartesian category C and for every object X in C, there is a unique strict cartesian functor $j : \mathbf{F^{op}} \to \mathbf{C}$ such that $j(\mathbf{s}) = \mathbf{X}$ where $\mathbf{s}$ is the new notation for the natural number 1 considered as an object of $\mathbf{F}$. More generally, by going from the cocartesian category $\mathbf{F}$ to the cartesian category $\mathbf{F^{op}}$, it will be convenient to change the notations, and in particular the monoidal structure $(\mathbf{F}, \mathbf{0}, +)$ will be written $(\mathbf{F^{op}}, \mathbf{1}, \times)$ (not to be confused with the cartesian structure of $\mathbf{F}$, which plays no role here). So, writing $\mathbf{s^n}$ for the object of $\mathbf{F^{op}}$ corresponding with the natural number n, we have $\mathbf{s^n} = \mathbf{s} \times \mathbf{s} \cdots \mathbf{s}$ (n times). Also the 2-cells $\eta : 0 \to 1$, $\mu : 2 \to 1$ and $\tau : 2 \to 2$ in $\mathbf{F}$ are respectively written $\varepsilon : \mathbf{s} \to \mathbf{1}$, $\delta : \mathbf{s} \to \mathbf{s^2}$ and $\sigma : \mathbf{s^2} \to \mathbf{s^2}$ in $\mathbf{F^{op}}$. The translation of equations (1) to (10) is left to the reader.

For any algebraic theory T (in the sense of Lawvere), the universal property of $\mathbf{F^{op}}$ gives a strict cartesian functor $i_T : \mathbf{F^{op}} \to \mathbf{T}$ such that $i_T(\mathbf{s}) = \mathbf{s}$ (we shall use the same notation for the cells in $\mathbf{F^{op}}$ and their image by i_T).

Theorem 2 *For all finite* (Ω, E)*, the 2-monoid* $T = \mathbf{T}(\boldsymbol{\Omega}, \mathbf{E})$ *is finitely presented (i.e. it has a finite CW-presentation in the sense of 1.2).*

The CW-presentation of T will consist of three packages. The first package is the one used for the presentation of $\mathbf{F}$, but dualized, and describes the fragment of T which is isomorphic to $\mathbf{F^{op}}$. The strict cartesian structure of $\mathbf{F^{op}}$ gives in particular, for every $n \in \mathbf{N}$, two morphisms

$$1 \xleftarrow{\varepsilon_n} \mathbf{s^n} \xrightarrow{\delta_n} \mathbf{s^{2n}}$$

defining two natural transforms and satisfying the conditions of proposition 1.

The second one is the obvious translation of (Ω, E) in terms of CW-presentation. Those two packages lead to the construction of a 2-monoid C and an epimorphism $\pi : C \to T$. The proof of the theorem would be complete if π were an isomorphism, that is if one could build a surjective section of π. Unfortunately, it is not possible to use the universal property of T here, because C is not necessarily cartesian, despite the presence of candidates ε_n and δ_n for natural transformations satisfying the conditions of proposition 1.

The aim of the third package, consisting only of collapsings, is to ensure the naturality of ε_n and δ_n (and not only in $\mathbf{F^{op}}$). In other words, for every $\lambda : \mathbf{s^p} \to \mathbf{s^q}$, the following equations must hold:

$$(I) \quad \varepsilon_q \circ \lambda = \varepsilon_p, \qquad (II) \quad \delta_q \circ \lambda = (\lambda \times \lambda) \circ \delta_p.$$

The rest of this section is devoted to showing how these conditions boil down to a finite number of collapsings. Precisely, for every generator $\alpha : \mathbf{s^n} \to \mathbf{s}$ in Ω (and there is only a finite number of them), we add the following collapsings:

$$(i) \quad \varepsilon \circ \alpha = \varepsilon_n, \qquad (ii) \quad \delta \circ \alpha = (\alpha \times \alpha) \circ \delta_n, \qquad (iii) \quad \sigma \circ (\alpha \times \mathbf{s}) = (\mathbf{s} \times \alpha) \circ \sigma_{\mathbf{n},\mathbf{1}},$$

where $\sigma_{n,1}$ corresponds to a "circular permutation". More precisely, for all $p, q \in \mathbf{N}$, we introduce $\sigma_{p,q} : \mathbf{s^{p+q}} \to \mathbf{s^{p+q}}$ in $\mathbf{F^{op}}$, dual of the map $\tau_{p,q} : p+q \to p+q$ in $\mathbf{F}$, defined by

$$\tau_{p,q}(i) = \begin{cases} p+i & \text{if } 0 \leq i < p, \\ i-p & \text{if } p \leq i < p+q. \end{cases}$$

Note that the following equation is a consequence of (iii) and the fact that $\sigma_{p,q}$ is involutive:

$$(iii') \quad \sigma \circ (\mathbf{s} \times \alpha) = (\alpha \times \mathbf{s}) \circ \sigma_{\mathbf{1},\mathbf{n}}$$

Note also that condition (iii) holds in any strict cartesian category, and more generally in a *symmetric* monoidal category. It is indeed a special case of the relation expressing the naturality of $\sigma_{p,q}$. From (iii), we derive:

$$(III) \quad \sigma_{q,1} \circ (\mathbf{s} \times \lambda) = (\lambda \times \mathbf{s}) \circ \sigma_{\mathbf{1},\mathbf{p}}$$

for all $\lambda : \mathbf{s^p} \to \mathbf{s^q}$. This is proved by induction on λ (saturating by the two compositions on 2-cells). From (III), we get

$$(IV) \quad \sigma_{q,q'} \circ (\lambda \times \lambda') = (\lambda' \times \lambda) \circ \sigma_{p,p'}$$

for all $\lambda : \mathbf{s^p} \to \mathbf{s^q}$ and $\lambda' : \mathbf{s^{p'}} \to \mathbf{s^{q'}}$. This follows directly from the following generalization of (III)

$$\sigma_{q,r} \circ (\lambda \times \mathbf{s^r}) = (\mathbf{s^r} \times \lambda) \circ \sigma_{\mathbf{p},\mathbf{r}}$$

which is proved by induction on $r \in \mathbf{N}$. Godement's rule is the basic ingredient for those calculations. In practise, the author uses directly a 2-dimensional language ("Penrose diagrams", see [Laf]) which make calculations much more transparent than with traditional (1-dimensional) categorical diagrams.

Finally, we prove relations (I) and (II), again by induction on λ. The calculations use the following relations which hold in $\mathbf{F^{op}}$:

$$\varepsilon_{p+p'} = \varepsilon_p \times \varepsilon_{p'} \qquad \delta_{p+p'} = (\mathbf{s^p} \times \sigma_{\mathbf{p},\mathbf{p'}} \times \mathbf{s^{p'}}) \circ (\delta_{\mathbf{p}} \times \delta_{\mathbf{p'}}).$$

By proposition 1, the presentation given by the three packages defines a 2-monoid C' which is a strict cartesian category, and a strict cartesian functor $\pi' : C' \to T$. Now the universal property of T gives a section of π' which is easily seen to be surjective. Therefore π' is an isomorphism. **Q.e.d.**

Finally, we illustrate how equations are translated into the previous system in the case of the theory of rings, limiting ourselves to the following distributivity axioms:

$$\forall x, y, z \quad x(y+z) = xy + xz, \qquad \forall x \quad x0 = 0.$$

ε, δ and σ allow the "management" of arities instead of the traditional variables (or the well-known notions of weakening, contraction and exchange in sequent calculus):

$$m \circ (\mathbf{s} \times \mathbf{a}) = \mathbf{a} \circ (\mathbf{m} \times \mathbf{m}) \circ (\mathbf{s} \times \sigma \times \mathbf{s}) \circ (\delta \times \mathbf{s^2}), \qquad \mathbf{m} \circ (\mathbf{s} \times \mathbf{z}) = \mathbf{z} \circ \varepsilon,$$

where $z : 1 \to \mathbf{s}$, $a : \mathbf{s^2} \to \mathbf{s}$ and $m : \mathbf{s^2} \to \mathbf{s}$ denote respectively zero, addition and multiplication. To understand the conceptual simplification, an appropriate symbolism is needed. We urgently refer the reader to this symbolism presented in [Laf] and we invite him to compare it with traditional representations in term rewriting.

Remark 1. In order to do computations in a T-algebra $F : T \to \mathbf{Set}$ (and not only in the theory T) is suffices to make computations in the theory T_F which is obtained through the adjoining of the elements of F (*i.e.* of $F(\mathbf{s})$) as constants, and through the relations they satisfy. At last, if F is finitely presented, it is sufficient to add the finite data corresponding to this presentation: T_F will still be finitely presented (whenever T is).

Remark 2. This result extend to many-sorted algebraic theories. The part played by $\mathbf{F}$ will be played by $\mathbf{F^I}$, I being the set of sorts.

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