AN INTRODUCTION TO PERVERSE SHEAVES AND CHARACTER SHEAVES

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Abstract. After a brief review of derived categories, triangulated categories and t-structures, we shall consider the bounded derived category $D^b(X, R)$ of the category of étale sheaves of R-modules on X (a k-scheme of finite type), where R is an ℓ-torsion ring with ℓ a prime number distinct from the characteristic of the field k. We shall construct a t-structure on the (triangulated) full subcategory $D_c^b(X, R)$ of $D^b(X, R)$ consisting of constructible complexes of sheaves. The category of perverse (with respect to the middle perversity function) sheaves $\mathcal{M}(X, R)$ will be defined as the heart of that t-structure.

Then we shall pass to the construction of $D_c^b(X, \bar{\mathbb{Q}}_\ell)$, the bounded "derived" category of $\bar{\mathbb{Q}}_\ell$-(constructible) sheaves on an algebraic variety X over an algebraically closed field in which the prime number ℓ is invertible [BBD, 2.2.18] and describe the corresponding subcategory $\mathcal{M}(X)$ of perverse sheaves on X.

In the second half, we shall define, and study some of the main properties of character sheaves over G, a reductive algebraic group defined over an algebraically closed field. Character sheaves on G, which were introduced by Lusztig in [Lu3] for a connected G (and extended by him more recently to disconnected G in [Lu8]), belong to the set of isomorphism classes of simple objects of $\mathcal{M}(G)$.

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Part 1. Perverse Sheaves

Perverse sheave have become a tool of great importance in representation theory, largely because of the remarkable way in which they provide a link between algebra and geometry. On one side, the category of perverse sheaves behaves like the modules categories typically seen in representation theory. On the other side, a single perverse sheaf contains information reminiscent of classical algebraic topology.

1. Categories of complexes

Definition 1.1. An **additive category** is a category $\mathcal{A}$ satisfying the following three axioms:

(1) For any two objects A and B in $\mathcal{A}$, the set of morphisms $\mathrm{Hom}(A, B)$ is endowed with the structure of an abelian group, and composition of morphisms is biadditive.
(2) There is an object 0 (called a "zero object") in $\mathcal{A}$ with the property that $\mathrm{Hom}(0, 0) = 0$. (This property implies that the object 0 is unique up to unique isomorphism, and that $\mathrm{Hom}(0, A) = \mathrm{Hom}(A, 0) = 0$ for any object A in $\mathcal{A}$.)
(3) For any two objects A and B in $\mathcal{A}$, there is a "biproduct" $A \oplus B$. It is equipped with morphisms

$$A \xrightarrow{i_1} A \oplus B \xleftarrow{i_2} B \quad \text{and} \quad A \xleftarrow{p_1} A \oplus B \xrightarrow{p_2} B,$$

which satisfy the following identities:

$$p_1 \circ i_1 = \mathrm{id}_A,\ \ p_2 \circ i_2 = \mathrm{id}_B,\ \ p_1 \circ i_2 = 0,\ \ p_2 \circ i_1 = 0,\ \ i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A\oplus B}.$$

An **abelian category** is an additive category $\mathcal{A}$ which satifies in addition the following axiom:

(4) For any two objects A and B in $\mathcal{A}$, every morphism $\phi\colon A \to B$ gives rise to a diagram

$$K \xrightarrow{k} A \xrightarrow{i} I \xrightarrow{j} B \xrightarrow{c} C,$$

where K is the kernel of ϕ, C is its cokernel, I is both the cokernel of $k\colon K \to A$ and the kernel of $c\colon B \to C$, and $j \circ i = \phi$. Such a diagram is called the **canonical decomposition** of ϕ. The object I is called the **image** of ϕ.

Example 1.2. Let $\mathfrak{Ab}$ denote the category of abelian groups. It is an abelian category.

Definition 1.3. A **simple** (or **irreducible**) **object** A in an abelian category is an object with the property that any monomorphism $A' \to A$ is either 0 or an isomorphism. Equivently, any epimorphism $A \to A''$ must be either 0 or an isomorphism.

Definition 1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be functors. We say that F is **left-adjoint** to G and that G is **right-adjoint** to F, or simply that (F, G) is an **adjoint pair**, if

$$\mathrm{Hom}_{\mathcal{B}}(F(A), B)) \simeq \mathrm{Hom}_{\mathcal{A}}(A, G(B)), \quad \text{for all objects } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Example 1.5. Let $\mathcal{A} = \mathcal{B} =$ the category of abelian groups. For a fixed abelian group C, the functors $\cdot \otimes C$ and $\mathrm{Hom}(C, \cdot)$ form an adjoint pair:

$$\mathrm{Hom}(A \otimes C, B) \simeq \mathrm{Hom}(A, \mathrm{Hom}(C, B)).$$

Definition 1.6. Let $\mathcal{A}$ be an abelian category. A **(cohomological) complex** of objects of $\mathcal{A}$ (or a **complex of cochains** in $\mathcal{A}$)

$$A^\bullet\colon \quad \cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots$$

is a sequence $A^\bullet$ of objects (A^i) labelled by integers, together with morphisms (called **differentials**)

$$d_A^i = d^i\colon A^i \to A^{i+1}, \quad \text{such that } d^i \circ d^{i-1} = 0.$$

The **category of complexes** over $\mathcal{A}$, denoted $C(\mathcal{A})$, is the category whose objects are complexes of object of $\mathcal{A}$, and in which a morphism $f\colon A^\bullet \to B^\bullet$ is a family of morphisms in $\mathcal{A}$, $f^i\colon A^i \to B^i$, that are compatible with the differentials, that is, every square in the following diagram should commute:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & A^{i-1} & \longrightarrow & A^i & \longrightarrow & A^{i+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & \\
\cdots \longrightarrow & B^{i-1} & \longrightarrow & B^i & \longrightarrow & B^{i+1} & \longrightarrow \cdots
\end{array}$$

Definition 1.7. The ith **cohomology functor** $\mathcal{H}^i\colon C(\mathcal{A}) \to \mathcal{A}$ is defined as follows: given an object $A^\bullet$ in $C(\mathcal{A})$, the natural morphism $\mathrm{im}\, d^{i-1} \to A^i$ factors through the kernel of d^i (by the universal property of the kernel, and the fact that $d^i \circ d^{i-1} = 0$), then $\mathcal{H}^i(A^\bullet)$ is defined to be the cokernel of the induced morphism $\mathrm{im}\, d^{i-1} \to \ker d^i$, that is

$$\mathcal{H}^i(A^\bullet) := \ker d^i / \mathrm{im}\, d^{i-1}.$$

A morphism $f\colon A^\bullet \to B^\bullet$ induces a morphism $\mathcal{H}^i(f)\colon \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet)$, so $\mathcal{H}^i$ is indeed a functor.

Definition 1.8. A **homotopy** between two morphisms $f, g\colon A^\bullet \to B^\bullet$ in $C(\mathcal{A})$ is a collection of morphisms $h^i\colon A^i \to B^{i-1}$, such that $d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i = f^i - g^i$ for each i. If there is a homotopy between f and g, then f and g are said to be **homotopic**.

Lemma 1.9. *If $f, g\colon A^\bullet \to B^\bullet$ are two homotopic morphisms in $C(\mathcal{A})$, then $\mathcal{H}^i(f) = \mathcal{H}^i(g)$.*

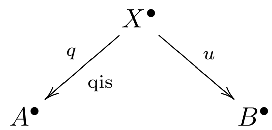
Definition 1.10. The **homotopy category** over $\mathcal{A}$, denoted $K(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, but whose morphisms are given by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/(\text{morphisms homotopic to } 0).$$

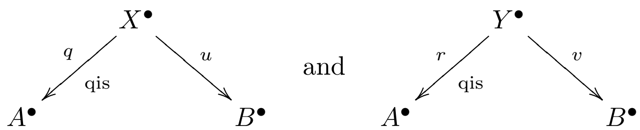
In other words, a morphism in $K(\mathcal{A})$ is a homotopy class of morphisms in $C(\mathcal{A})$.

Definition 1.11. A morphism $f\colon A^\bullet \to B^\bullet$ (in the category $C(\mathcal{A})$ or $K(\mathcal{A})$) is a **quasi-isomorphism** if the morphisms $\mathcal{H}^i(f)$ in $\mathcal{A}$ are isomorphisms for all i.

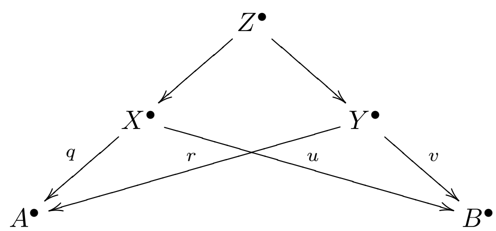
Definition 1.12. The **derived category** of $\mathcal{A}$, denoted $D(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, and in which a morphism from $A^\bullet$ to $B^\bullet$ is an equivalence class of "roofs"



where q is a quasi-isomorphism. The equivalence relation is as follows: Two roofs



are equivalent if there exists a roof over $X^\bullet$ and $Y^\bullet$ making the following diagram commute:



Definition 1.13. Let $n \in \mathbb{Z}$, and let $A^\bullet$ be a complex of objects of $\mathcal{A}$. The nth **translation** (or **shift**) is the complex $A[n]^\bullet$ defined by

$$A[n]^i := A^{i+n} \quad \text{with differential} \quad d^i_{A[n]^\bullet} := (-1)^n d^{i+n}_{A^\bullet}.$$

Given a morphism $f\colon A^\bullet \to B^\bullet$, the **translated morphism** $f[n]\colon A[n]^\bullet \to B[n]^\bullet$ is defined by $f[n]^i := f^{i+n}$. In this way, translation by n is a functor $T\colon C(\mathcal{A}) \to C(\mathcal{A})$ (or $K(\mathcal{A}) \to K(\mathcal{A})$, or $D(\mathcal{A}) \to D(\mathcal{A})$).

Definition 1.14. Let $f\colon A^\bullet \to B^\bullet$ be a morphism in $C(\mathcal{A})$. The **cone** of f, denoted $\mathrm{cone}^\bullet f$, is the complex defined as follows:

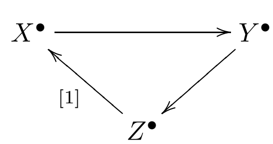
$$\mathrm{cone}^i f = A^{i+1} \oplus B^i,$$

with differential $d^i_{\mathrm{cone}^\bullet f}\colon A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}$ given by $\begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$.

There are natural morphisms

$$\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} : B^\bullet \to \mathrm{cone}^\bullet f \quad \text{and} \quad \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} : \mathrm{cone}^\bullet f \to A[1]^\bullet.$$

Definition 1.15. A sequence of morphisms $X^\bullet \to Y^\bullet \to Z^\bullet \to X[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ is called a **triangle**. It will be sometimes written as



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The triangle is **distinguished** if there is a commutative diagramm

$$\begin{array}{ccccccc}
X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X[1]^\bullet \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle q[1]} \\
A^\bullet & \xrightarrow[f]{} & B^\bullet & \longrightarrow & \mathrm{cone}^\bullet(f) & \longrightarrow & A[1]^\bullet
\end{array}$$

where a, r and s are isomorphisms. (Of course, the meaning of "isomorphisms", and hence of "distinguished triangle", depends on wether one is working in $K(\mathcal{A})$ or $D(\mathcal{A})$.)

Theorem 1.16. *Distinguished triangles in $K(\mathcal{A})$ enjoy the following four properties:*

(1) **(Identity)** *The triangle $A^\bullet \xrightarrow{\mathrm{id}} A^\bullet \to 0 \to A[1]^\bullet$ is distinguished.*
(2) **(Rotation)** *The triangle $A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet \to A[1]^\bullet$ is distinguished if and only if the* **rotated** *triangle $B^\bullet \to C^\bullet \to A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet$ is.*
(3) **(Square Completion)** *Any commutative square*

$$\begin{array}{ccc}
A^\bullet & \longrightarrow & B^\bullet \\
\downarrow{\scriptstyle u} & & \downarrow \\
F^\bullet & \longrightarrow & G^\bullet
\end{array}$$

can be completed to a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & A[1]^\bullet \\
\downarrow{\scriptstyle u} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle u[1]} \\
F^\bullet & \longrightarrow & G^\bullet & \longrightarrow & H^\bullet & \longrightarrow & F[1]^\bullet
\end{array} \ .$$

(4) **(Octahedral Property)** *see* [BBD, §1.1.6]

Theorem 1.17. *Distinguished triangles in $D(\mathcal{A})$ enjoy the following five properties:*

(0) **(Existence)** *Any morphism $A^\bullet \to B^\bullet$ can be completed into a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$.*
(1–4) *The Identity, Rotation, Square Completion, and Octahedral Property of distinguished triangles in $K(\mathcal{A})$.*

Proposition 1.18. *A distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ gives rise to a long exact sequence in cohomology*

$$\cdots \to \mathcal{H}^{i-1}(C^\bullet) \to \mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet) \to \mathcal{H}^i(C^\bullet) \to \mathcal{H}^{i+1}(A^\bullet) \to \cdots .$$

The derived category $D(\mathcal{A})$ (resp. the homotopy category $K(\mathcal{A})$) of an abelian category $\mathcal{A}$, with its shift functor [1] and its notion of distinguished triangles is a **triangulated category**:

2. Triangulated Categories and **t**-Structures

Definition 2.1. A **triangulated category** is an additive category $\mathcal{C}$ equipped with a **shift functor** $[1]\colon \mathcal{C} \to \mathcal{C}$, $X \mapsto X[1]$, and a class of sequences (called **distinguished triangles**) $X \to Y \to Z \to X[1]$, satisfying the following seven axioms:

(-2) The shift functor is an autoequivalence of $\mathcal{C}$. In particular, it has an inverse, denoted $[-1]\colon \mathcal{C} \to \mathcal{C}$.
(-1) Every triangle isomorphic to a distinguished triangle is a distinguished triangle.
(0) (Existence) Every morphism $f\colon X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.
(1) (Identity).
(2) (Rotation),
(3) (Square Completion),
(4) (Octahedral Property).

Remark 2.2. Note that "being abelian" is a property of an additive category, whereas "being triangulated" is the datum of extra structure.

Definition 2.3. A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is **strictly full** if
(a) For any pair of objects (A, B) in $\mathcal{C}'$, we have $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$;
(b) Any object A of $\mathcal{C}$ which is isomorphic to an object of C$'$ is an object of C$'$.

Definition 2.4. Let $\mathcal{C}$ be a triangulated category. A **t-structure** on $\mathcal{C}$ is a pair of strictly full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ satisfying the four axioms below. (For any $n \in \mathbb{Z}$, we use the notation $\mathcal{C}^{\leq n} := \mathcal{C}^{\leq 0}[-n]$ and $\mathcal{C}^{\geq n} := \mathcal{C}^{\geq 0}[-n]$.)
(1) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 0} \supset \mathcal{C}^{\geq 1}$.
(2) $\bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\leq n} = \bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\geq n} = \{0\}$.
(3) If $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$, then $\mathrm{Hom}(A, B) = 0$.
(4) For any object X in $\mathcal{C}$, there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$.

The full subcategory $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\leq 0}$ is called the **heart** (or **core**) of the t-structure.

Proposition 2.5. *The objects A and B in the distinguished triangle in Axiom (4) in Definition 2.4 are unique up to unique isomorphism and they depend functorially in X.*

Indeed, there are functors (called **truncation functors***)*

$$ {}^t\tau_{\leq 0}\colon \mathcal{C} \to \mathcal{C}^{\leq 0} \quad \text{and} \quad {}^t\tau_{\geq 1}\colon \mathcal{C} \to \mathcal{C}^{\geq 1}, $$

such that for any object X of $\mathcal{C}$,

$$ {}^t\tau_{\leq 0}X \to X \to {}^t\tau_{\geq 1}X \to ({}^t\tau_{\leq 0}X)[1] $$

is that distinguished triangle.

For any $n \in \mathbb{N}$, let ${}^t\tau_{\leq n}\colon \mathcal{C} \to \mathcal{C}^{\leq n}$ and ${}^t\tau_{\geq n+1}\colon \mathcal{C} \to \mathcal{C}^{\geq n+1}$ be the shifted truncation functors to be defined by:

$$ {}^t\tau_{\leq n} := T^{-n} \circ {}^t\tau_{\leq 0} \circ T^n \quad \text{and} \quad {}^t\tau_{\geq n+1} := T^{-n} \circ {}^t\tau_{\geq 1} \circ T^n, $$

that is,

$$ {}^t\tau_{\leq n}X = ({}^t\tau_{\leq 0}(X[n]))[-n] \quad \text{and} \quad {}^t\tau_{\geq n+1}X = ({}^t\tau_{\geq 1}(X[n]))[-n] $$

for any object X of $\mathcal{C}$.

Proposition 2.6. *Let $\iota_{\leq n}\colon \mathcal{C}^{\leq n} \to \mathcal{C}$ be the inclusion functor. Then $(\iota_{\leq n}, {}^t\tau_{\leq n})$ is an adjoint pair (see Definition 1.4). Similarly, let $\iota_{\geq n+1}\colon \mathcal{C}^{\geq n+1} \to \mathcal{C}$ be the inclusion functor. Then $({}^t\tau_{\geq n+1}, \iota_{\geq n+1})$ is an adjoint pair.*

Proposition 2.7. *Any distinguished triangle* $A \to B \to C \to A[1]$ *where* A *is an object of* $\mathcal{C}^{\leq n}$ *and* C *is an object of* $\mathcal{C}^{\geq n+1}$ *is canonically isomorphic to the distinguished triangle*

$$^t\tau_{\leq n} B \to B \to {}^t\tau_{\geq n+1} B \to ({}^t\tau_{\leq n} B)[1].$$

Remark 2.8. Proposition 2.7 is a strong statement. It shows the uniqueness of the shifted truncation functors. It also proves that the following equivalences hold:

(1) A is an object of $\mathcal{C}^{\leq n}$ if and only if $A = {}^t\tau_{\leq n} A$.
(2) A is an object of $\mathcal{C}^{\geq n+1}$ if and only if $A = {}^t\tau_{\geq n+1} A$.

Proposition 2.9. *The heart* $\mathcal{T} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ *of the* t*-structure* $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ *is an* **admissible** *abelian subcategory of* $\mathcal{C}$*, i.e., an abelian subcategory with the following property:*

> *For any short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* $\mathcal{C}$*, there exists a distinguished triangle*
>
> $$A \xrightarrow{f} B \xrightarrow{g} C \to A[1]$$
>
> *in* $\mathcal{C}$*.*

Moreover, the heart $\mathcal{T}$ *is stable by triangulated extensions.*

Definition 2.10. The functor ${}^t\mathcal{H}^0 \colon \mathcal{C} \to \mathcal{T}$ defined by

$$^t\mathcal{H}^0 := {}^t\tau_{\leq 0}{}^t\tau_{\geq 0} = {}^t\tau_{\geq 0}{}^t\tau_{\leq 0}$$

is called the **(zeroth) t-cohomology** functor. More generally, for any $i \in \mathbb{Z}$, the functor ${}^t\mathcal{H}^i \colon \mathcal{C} \to \mathcal{T}$ defined by

$$^t\mathcal{H}^i := {}^t\tau_{\leq i}{}^t\tau_{\geq i} = {}^t\tau_{\geq i}{}^t\tau_{\leq i}$$

(*i.e.*,

$$^t\mathcal{H}^i(X) = {}^t\mathcal{H}^0(X[i]) \quad \text{for any object } X \text{ of } \mathcal{C})$$

is called the **(ith) t-cohomology** functor.

The following theorem is the reason we want to introduce the notion of t-structures.

Theorem 2.11. *The functors* ${}^t\mathcal{H}^i$ *satisfy the following properties:*

(1) *For any object* A *of* $\mathcal{T}$*, we have* ${}^t\mathcal{H}^0(A) \simeq A$.
(2) *The functor* ${}^t\mathcal{H}^0$ *takes distinguished triangles in* $\mathcal{C}$ *to long exact sequences in* $\mathcal{T}$*. (Such a functor is said to be* **cohomological***.)*
(3) *A morphism* $f \colon X \to Y$ *in* $\mathcal{C}$ *is an isomorphism if and only if the morphisms* ${}^t\mathcal{H}^i(f)$ *are isomorphisms in* $\mathcal{T}$ *for all* $i \in \mathbb{Z}$.
(4) *We have*

$$\mathcal{C}^{\leq 0} = \left\{X \text{ object of } \mathcal{C} \,:\, {}^t\mathcal{H}^i(X) = 0 \text{ for all } i > 0\right\},$$
$$\mathcal{C}^{\geq 0} = \left\{X \text{ object of } \mathcal{C} \,:\, {}^t\mathcal{H}^i(X) = 0 \text{ for all } i < 0\right\}.$$

Example 2.12. Let $\mathcal{A}$ be an abelian category, and let $D(\mathcal{A})$ denote its derived category. One defines a t-structure (called the **natural t-structure**) on $D(\mathcal{A})$ by setting:

$$D(\mathcal{A})^{\leq 0} := \left\{A^\bullet \in D(\mathcal{A}) \,:\, \mathcal{H}^i A^\bullet = 0 \;\; \forall i > 0\right\},$$
$$D(\mathcal{A})^{\geq 0} := \left\{A^\bullet \in D(\mathcal{A}) \,:\, \mathcal{H}^i A^\bullet = 0 \;\; \forall i < 0\right\},$$

with the following truncation functors:

$$\tau_{\leq 0}A^\bullet := \left(\cdots \to A^{-2} \overset{d^{-2}}{\to} A^{-1} \overset{d^{-1}}{\to} \ker d^0 \to 0\right)$$
$$\tau_{\geq 0}A^\bullet := \left(0 \to \operatorname{coker} d^{-1} \overset{d^{0}}{\to} A^{1} \overset{d^{1}}{\to} A^2 \to \cdots\right).$$

The heart of this t-structure is $\mathcal{A}$ itself, and we have ${}^t\mathcal{H}^i = \mathcal{H}^i$ for any i.

Definition 2.13. Let F be a functor between two triangulated categories $\mathcal{C}$ and $\mathcal{C}'$, itself equipped with a t-structure. The functor F is said to be **left t-exact** if $F(\mathcal{C}^{\geq 0}) \subset \mathcal{C}'^{\geq 0}$, and to be **right t-exact** if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{C}'^{\leq 0}$.

3. Derived Categories and Derived functors

Definition 3.1. An object $A^\bullet$ of the category $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, $D(\mathcal{A})$) is called **cohomologically bounded-below** if there exists an integer $N = N_{A^\bullet}$ (which may depend on $A^\bullet$) such that

$$\mathcal{H}^i(A^\bullet) = 0 \quad \text{for all } i < N.$$

The **cohomologically bounded-above** objects are defined similarly.

An object is called **cohomologically bounded** if it is both cohomologically bounded-below and cohomologically bounded-above.

The **bounded-below category of complexes** (resp. **bounded-below homotopy category**, **bounded-below derived category**) over $\mathcal{A}$ is the full subcategory $C^+(\mathcal{A})$ (resp. $K^+(\mathcal{A})$, $D^+(\mathcal{A})$) of $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, $D(\mathcal{A})$) containing those objects which are cohomologically bounded-below. The **bounded-above** categories $C^-(\mathcal{A})$, $K^-(\mathcal{A})$, $D^-(\mathcal{A})$, and the **bounded** categories $C^b(\mathcal{A})$, $K^b(\mathcal{A})$, $D^b(\mathcal{A})$ are defined similarly.

Definition 3.2. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is **large enough** (or one says that $\mathcal{A}$ **has enough objects in** $\mathcal{R}$) if for every object A of $\mathcal{A}$, there is a monomorphism $A \to R$, where $R \in \mathcal{R}$.

Theorem 3.3. (Existence of Resolutions) *If $\mathcal{R}$ is large enough, then for any complex $A^\bullet \in D^+(\mathcal{A})$, there exists a complex $R^\bullet \in D^+(\mathcal{A})$ with $R^i \in \mathcal{R}$ for all i, and a quasi-isomorphism $t\colon A^\bullet \to R^\bullet$. The complex $R^\bullet$ is called an* $\mathcal{R}$**-resolution** *of $A^\bullet$.*

Definition 3.4. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called

- **exact** if F takes any short exact sequence of objects in $\mathcal{A}$ to a short exact sequence in $\mathcal{B}$;
- **left-exact** if for each exact sequence

  $$0 \to A \to A' \to A'' \quad \text{where } A,\ A',\ A'' \text{ are objects in } \mathcal{A}$$

  the sequence

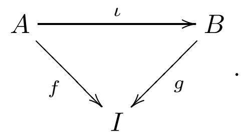
  $$0 \to F(A) \to F(A') \to F(A'')$$

  is exact.

Definition 3.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is an **adapted class of objects** for F if the following two conditions are satisfied

1. F is exact on $\mathcal{R}$ (that is, F takes any short exact sequence of objects in $\mathcal{R}$ to a short exact sequence in $\mathcal{B}$).
2. $\mathcal{R}$ is large enough.

Definition 3.6. An object I of an abelian category $\mathcal{A}$ is **injective** if, for every morphism $f\colon A \to I$ and every monomorphism $\iota\colon A \hookrightarrow B$, there exists a morphism $g\colon B \to I$ making the following diagram commute:



Proposition 3.7. *Any left-exact functor is exact on the class $\mathcal{I}$ of injective objects.*

Corollary 3.8. *If $\mathcal{A}$ has enough injectives, the the class $\mathcal{I}$ of injective objects is an adapted class for every left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$.*

Definition 3.9. (Derived functor) Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor, and let $\mathcal{R}$ be an adapted class for F. The **derived functor**

$$RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

is defined as follows: for any object $A^\bullet$ in $D^+(A)$, choose an $\mathcal{R}$-resolution $R^\bullet$, and let $RF(A^\bullet)$ be the complex $F(R^\bullet)$. The latter is well-defined up to quasi-isomorphism (*i.e.,* up to isomorphism in $D(\mathcal{B})$) because $R^\bullet$ is unique up to quasi-isomorphism, and since F is exact on $\mathcal{R}$, it takes quasi-isomorphisms of complexes of objects in $\mathcal{R}$ to quasi-isomorphisms.

Remark 3.10. It can be shown that if $\mathcal{R}$ and $\mathcal{R}'$ are two adapted classes for F, they both give rise to the same notion of derived functor. In the examples we shall meet, we shall only work with adapted classes that contains a fixed adapted class $\mathcal{R}_0$ (specifically, $\mathcal{R}_0$ will be the class of injective objects). Now, if $\mathcal{R} \supset \mathcal{R}_0$ then every $\mathcal{R}_0$-resolution is also an $\mathcal{R}$-resolution, so the derived functors defined with respect to $\mathcal{R}$ and to $\mathcal{R}_0$ coincide.

Definition 3.11. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor. The **ith classical derived functor**

$$R^iF\colon \mathcal{A} \to \mathcal{B}$$

is defined as follows: given an object A of $\mathcal{A}$, regard it as a complex in $D(\mathcal{A})$ with a single nonzero term located in degree 0. Then

$$R^iF(A) := \mathcal{H}^i(RF(A)).$$

Example 3.12. Let A be an object of $\mathcal{A}$. The functor $\operatorname{Hom}(A, -)\colon \mathfrak{Ab} \to \mathfrak{Ab}$ is left-exact, and,if the category $\mathcal{A}$ has enough injectives (for instance for $\mathcal{A} = \mathfrak{Ab}$), it gives rise to a derived functor $R\operatorname{Hom}\colon D^+(\mathcal{A}) \to D^+(\mathcal{A})$. The corresponding classical derived functors $R^i \operatorname{Hom}(A, -)$ are usually denoted $\operatorname{Ext}^i(A, -)$.

Definition 3.13. An object A of $\mathcal{A}$ is **F-(right-)acyclic** if, $R^iF(A) = 0$ for any $i > 0$.

Definition 3.14. A functor F is of **finite cohomological dimension** if there exists an integer n such that

$$R^iF(A) = 0 \quad \text{for any } i > n \text{ and any } A \in \mathcal{A}.$$

There is a parallel theory for right-exact functors which gives rise to left derived functors $LF\colon D^-(\mathcal{A}) \to D^-(\mathcal{B})$. In this setting, the class of *projectives* takes the place of the class of injectives. Then we set

$$L^iF(A) := \mathcal{H}^{-i}(LF(A)).$$

Example 3.15. If the category $\mathcal{A}$ has enough projectives (for instance when $\mathcal{A} = \mathfrak{Ab}$), then, for a fixed object A of $\mathcal{A}$, the right-exact functor $A \otimes -$ gives rise to a left derived functor $A \otimes^L -$. The classical derived functors known as "Tor" are defined by

$$\mathrm{Tor}_i(A, B) := \mathcal{H}^{-i}(A \otimes^L B).$$

Remark 3.16. The categories of great interest to us do not, in general, have enough projectives. For instance the category $\mathfrak{Sh}_{\mathrm{et}}(X, R)$ that we shall meet in Definition 4.6 does not, in general, have enough projectives. We shall try to replace the class of projectives by a convenient class of *(left) acyclic objects*.

Remark 3.17. The derived functors $R\,\mathrm{Hom}(-,-)$ and $- \otimes^L -$ are asymmetric in that they take an object of $\mathcal{A}$ in the first variable, but a complex from $D(\mathcal{A})$ in the second. It would be more natural to allow complexes in both variables. The right way to handle this problem is to develop a full theory of derived *bifunctors* (that is, functors of two variables). Such a theory is develop in [KS]. We shall not develop it here, but only show howe one can repair the definitions of $R\,\mathrm{Hom}(-,-)$ and $- \otimes^L -$ to allow complexes in both variables.

Definition 3.18. Let $A^\bullet$ and $B^\bullet$ be in $C(\mathcal{A})$.

(1) Let $\underline{\mathrm{Hom}}(A^\bullet, B^\bullet)$ (called "graded Hom") be the complex defined by

$$\underline{\mathrm{Hom}}(A^\bullet, B^\bullet)^i := \bigoplus_{k-j=i} \mathrm{Hom}(A^j, B^k),$$

where the differential $d\colon \underline{\mathrm{Hom}}(A^\bullet, B^\bullet)^i \to \underline{\mathrm{Hom}}(A^\bullet, B^\bullet)^{i+1}$ are viewed as collection of maps

$$(d)_{jk,mn}\colon \begin{array}{l} \mathrm{Hom}(A^j, B^k) \to \mathrm{Hom}(A^m, B^n) \\ f \mapsto \begin{cases} d_B^k \circ f & \text{if } m = j \text{ and } n = k+1, \\ (-1)^j f \circ d_A^{j-1} & \text{if } m = j-1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases} \end{array},$$

where $k - j = i$ and $n - m = i + 1$. (It can be easily checked that $d^2 = 0$, *i.e.,* that d makes $\underline{\mathrm{Hom}}(A^\bullet, B^\bullet)$ into a complex.)

(2) If $\mathcal{A}$ has enough injectives, then $R\,\mathrm{Hom}(A^\bullet, B^\bullet)$ is defined to be $\underline{\mathrm{Hom}}(A^\bullet, I^\bullet)$, where $I^\bullet$ is an injective resolution of $B^\bullet$.

Definition 3.19. Let $A^\bullet$ and $B^\bullet$ be in $C(\mathcal{A})$.

(1) Let $A^\bullet \underline{\otimes} B^\bullet$ (called "graded tensor product") be the complex defined by

$$(A^\bullet \underline{\otimes} B^\bullet)^i := \bigoplus_{j+k=i} A^j \otimes B^k,$$

where the differential $d\colon (A^\bullet \underline{\otimes} B^\bullet)^i \to (A^\bullet \underline{\otimes} B^\bullet)^{i+1}$ are viewed as collection of maps

$$(d)_{jk,mn}\colon \begin{array}{l} A^j \otimes B^k \to A^m \otimes B^n \\ f \mapsto \begin{cases} \mathrm{id}_{A^j} \otimes d_B^k & \text{if } m = j \text{ and } n = k+1, \\ (-1)^j d_A^j \otimes \mathrm{id}_{B^k} & \text{if } m = j+1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases} \end{array},$$

where $k + j = i$ and $n + m = i + 1$.

(2) If there exists a projective (as when $\mathcal{A} = \mathfrak{Ab}$), or *flat* (as in $\mathfrak{Sh}_{\text{et}}(X, R)$, see 4.11) resolution $P^\bullet$ of $B^\bullet$, then $A^\bullet \otimes^L B^\bullet$ is defined to be $A^\bullet \underline{\otimes} P^\bullet$.

Then we set

$$\operatorname{Tor}_i(A^\bullet, B^\bullet) := \mathcal{H}^{-i}(A^\bullet \otimes^L B^\bullet).$$

We shall say that $A^\bullet$ is of **finite** Tor**-dimension** if the functor $A^\bullet \otimes^L -$ is of finite cohomological dimension (see Definition 3.14).

4. Étale Topology

In this section we will fix a scheme X.

4.1. Definition of the category $\operatorname{Et}(X)$.

Definition 4.1. An **étale open set** on X is a pair (U, u) where U is a scheme and $u: U \to X$ is an étale morphism of finite type.

Let (U, u), (V, v) be two étale open sets on X. A **morphism of étale open sets** from (U, u) to (V, v) is any morphism of schemes $w: U \to V$ such that $u = v \circ w$.

A family $(U_i, u_i)_{i \in I}$ of étale open sets on X such that $X = \bigcup_{i \in I} u_i(U_i)$ is called an **étale covering** of X. (We shall often denote (U, u), (U_i, u_i) simply by U, U_i.)

Definition 4.2. Let $\operatorname{Et}(X)$ denote the category whose objects are the étale open sets on X and whose morphisms are the morphisms of étale open sets on X.

For R a ring, let Mod_R be the category of R-modules.

Definition 4.3. A contravariant functor

$$\mathcal{F}: \operatorname{Et}(X) \to \mathfrak{Ab} \quad (\text{resp. } \mathcal{F}: \operatorname{Et}(X) \to \operatorname{Mod}_R)$$

is called an étale presheaf of abelian groups (resp. of R-modules) on X.

Let $\mathcal{F}$ be an étale presheaf on X, let U be an étale open set on X, let V be an étale open set on U, and let $s \in \mathcal{F}(U)$. Then the image of s by the morphism $\rho_{VU}: \mathcal{F}(U) \to \mathcal{F}(V)$ is called the **restriction** of s to V: $s|_V := \rho_{VU}(s)$.

Let $(U_i)_{i \in I}$ be an étale covering of U, and $(s_i)_{i \in I}$ such that $s_i \in \mathcal{F}(U_i)$ for any $i \in I$. Then, for any $i, j \in I$, $U_i \times_U U_j$ is an étale open set on U_i . Let s_{ij} be the restriction of s_i to $U_i \times_U U_j$. We shall say that the family $(s_i)_{i \in I}$ satisfies the **gluing condition** if $s_{ij} = s_{ji}$ for any $i, j \in I$.

Definition 4.4. An étale presheaf of R-modules $\mathcal{F}$ on X is called an **étale sheaf** of R-modules if for each étale open set U on X, each étale covering $(U_i)_{i \in I}$ of U, and each family $(s_i)_{i \in I}$ which satisfies the gluing condition, there exists a unique $s \in \mathcal{F}(U)$ such that s_i is the restriction of s to U_i for any $i \in I$.

A morphism of functors between étale (pre)sheaves is called a **morphism of étale (pre)sheaves**.

Example 4.5. Let F be an abelian group, endowed with the discrete topology. For every étale open set U on X, let $\mathcal{F}(U)$ be the abelian group of all continuous maps from U to F (*i.e.,* all the functions which are constant on the connected components of U). Then $\mathcal{F}: \operatorname{Et}(X) \to \mathfrak{Ab}$ is an étale sheaf, called the **constant étale sheaf attached to** F.

Definition 4.6. Let $\mathfrak{Sh}_{\text{et}}(X, R)$ denote the category of étale sheaves of R-modules on X. The category $\mathfrak{Sh}_{\text{et}}(X, R)$ is abelian.

Definition 4.7. A **geometric point** of X is any morphism of schemes $\overline{x} \to X$, where $\overline{x}$ is the spectrum of an algebraically closed field. To simplify we shall often just say that $\overline{x}$ is a geometric point. The image of $\overline{x}$ will be denoted x.

Definition 4.8. An **étale neighbourhood** of $\overline{x}$, a geometric point (see Definition 4.7), is any pair (U, ι) where U is an open étale set on X and $\iota \colon \overline{x} \to U$ is a morphism of schemes which makes the following diagram commute:

$$\begin{array}{ccc} \overline{x} & \longrightarrow & X \\ & {\scriptstyle\iota}\searrow \quad \nearrow & \\ & U & \end{array} .$$

(We shall often just say that U is an étale neighbourhood of $\overline{x}$.)

Definition 4.9. Let $\mathcal{F}$ be an étale sheaf and let $\overline{x}$ be a geometric point. The **stalk** of $\mathcal{F}$ at $\overline{x}$, denoted $\mathcal{F}_{\overline{x}}$, is defined by

$$\mathcal{F}_{\overline{x}} := \varinjlim_{U \text{ étale neighbourhood of } \overline{x}} \mathcal{F}(U).$$

The notion of stalk of an étale sheaf allows us to define the **support** of the sheaf:

Definition 4.10. The **support** of an étale sheaf $\mathcal{F}$, denoted $\mathrm{Supp}(\mathcal{F})$, is defined to be the closure of the set

$$\{x \in X \,:\, \mathcal{F}_{\overline{x}} \neq 0\}.$$

Definition 4.11. An R-module M is **flat** if the functor $- \otimes_R M$ is exact.

A **flat** R-sheaf on X is an étale sheaf of R-modules $\mathcal{F}$ such that, for every geometric point $\overline{x}$ of X, the stalk $\mathcal{F}_{\overline{x}}$ is a flat R-module.

Every étale sheaf of R-modules on X is quotient of a flat R-sheaf on X.

4.2. **Functors between categories of étale sheaves.** Let X and Y be two schemes, and let $f \colon X \to Y$ be a scheme morphism. The aim of this subsection is to construct several functors between the categories $\mathfrak{Sh}_{\mathrm{et}}(X, R)$ and $\mathfrak{Sh}_{\mathrm{et}}(Y, R)$ which are induced by f.

Let V be an étale open set on Y, ane let $V \times_Y X$ denote the **fiber product** of V and X over Y (as defined in [Ha, Definition p.87]). Then $Y \times_X X$ is an étale open set on X.

Definition 4.12. If $\mathcal{F}$ is an étale sheaf of R-modules on X, the **push-forward** by f of $\mathcal{F}$, also called its **direct image**, and denoted $f_*\mathcal{F}$, is the étale sheaf of R-modules on Y defined by

$$(f_*\mathcal{F})(V) := \mathcal{F}(V \times_Y X).$$

Remark 4.13. The functor f_* generalizes the notion of push-forward for usual sheaves (as defined for instance in [Ha, Definition p.65]). Indeed, si V is an (ordinary) open set of Y, then $V \times_Y X$ is isomorphic to $f^{-1}(V)$.

Let U be an étale open set on X. Let $\mathcal{V}_U$ denote the set of étale open sets V on Y for which there exists a morphism $U \to V$ making the following diagram commute:

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array} .$$

Definition 4.14. If $\mathcal{F}$ is an étale sheaf of R-modules on X, the **pull-back** by f of $\mathcal{F}$, also called its **inverse image**, and denoted $f^*\mathcal{F}$ (or by $f^{-1}\mathcal{F}$), is the sheafification of the presheaf of R-modules $\mathrm{ps}f^*\mathcal{F}$ on Y defined by

$$(\mathrm{ps}f^*\mathcal{F})(U) := \varinjlim_{V \in \mathcal{V}_U} \mathcal{F}(V).$$

Remark 4.15. If the morphism f is étale of finite type, then $f^*\mathcal{F}$ is simply the **restriction** of $\mathcal{F}$ to X, *i.e.,* if U is an étale open set on X, then U is also an étale open set on Y and $(f^*\mathcal{F})(U) = \mathcal{F}(U)$. In this case we shall sometimes denote $f^*\mathcal{F}$ simply $\mathcal{F}|_X$.

Proposition 4.16. *The pair (f^*, f_*) is an adjoint pair.*

Proof. See for instance [Mi, Chap. II, Prop. 2.2]. □

Definition 4.17. Let $\mathcal{F}$ be an étale sheaf of R-modules on X. The **proper pusforward** by f of $\mathcal{F}$, denoted $f_!\mathcal{F}$, is the étale sheaf of R-modules on Y defined by

$$(f_!\mathcal{F})(V) := \left\{ s \in \mathcal{F}(V \times_Y X) : \; f|_{\mathrm{Supp}(s)} \colon \mathrm{Supp}(s) \to V \text{ is proper} \right\}.$$

(For the definition of a *proper morphism of schemes* see [Ha, Def. on page 100].)

Remark 4.18. If X is an open subset (in the classical sense) of Y, and $\iota_X \colon X \hookrightarrow Y$ denotes the canonical open immersion, then the functor $(\iota_X)_!$ is the **extension by zero** functor.

A fourth functor, denoted $f^!$, can be defined in the following special case. We assume that X is a subscheme of Y and that $f \colon X \to Y$ is an open immersion. We write X as $X = F \cap W$, where F (resp. W) is a closed (resp. open) étale subset of Y. Let $\mathcal{G}$ be an object of $\mathfrak{Sh}(Y, R)$ and $U \to X$ be an étale morphism, then we choose an étale morphism $V \to Y$ such that $U = V \times_Y F$.

Definition 4.19. Let $\mathcal{F}$ be an étale sheaf of R-modules on X. The **proper pullback** by f of $\mathcal{F}$, denoted $f^!\mathcal{F}$, is the étale sheaf of R-modules on Y defined by

$$(f^!\mathcal{G})(U) := \{ s \in \mathcal{G}(V) \,:\, \mathrm{Supp}\, s \subset V \times_Y X \}. \tag{1}$$

The RHS of (1) does not depend on the choices of F, W and V.

Proposition 4.20. *The pair $(f_!, f^!)$ is an adjoint pair.*

Proposition 4.21. *The functors f_*, $f_!$ and $f^!$ are left-exact. The functor f^* is exact. If f is proper, then we have $f_! = f_*$. If f is proper with finite fibres, then the functor $f_! = f_*$ is exact. If X is an étale open subset of Y, then $f^! = f^*$.*

Notations 4.22.
(1) If $U \xrightarrow{f} X$ is an étale scheme over X and $\mathcal{F}$ is an étale sheaf on X, we will set $\mathcal{F}_U := f_! f^* \mathcal{F}$.
(2) Let $f \colon X \to Y$ be a morphism between two algebraic varieties X and Y. If f is smooth with connected fibres of dimension d, then we have $f^! = f^*[2d](d)$, where (d) denotes Tate twist; in this case, we set $\tilde{f} := f^*[d]$.

4.3. **Constructible étale sheaves.**

Definition 4.23. Let $\mathcal{F}$ be an étale sheaf (of abelian groups) on X. Then $\mathcal{F}$ is called:

(1) **finite** if, for any quasi-compact étale open set U on X, the group $\mathcal{F}(U)$ is finite;
(2) **with finite stalks** if for each geometric point $\overline{x}$ on X the stalk $\mathcal{F}_{\overline{x}}$ of $\mathcal{F}$ at $\overline{x}$ is finite;
(3) **locally constant** if there is an étale covering $(U_i)_{i\in I}$ of X such that, for any $i \in I$, the étale sheaf $\mathcal{F}|_{U_i}$ on U_i is constant;
(4) **constructible** if, for any irreducible closed subscheme Z of X, there exists a non-empty open set U of Z such that $\iota_U^*(\mathcal{F})$ is locally constant with finite stalks (where $\iota_U : U \to X$).

Remark 4.24. If $\mathcal{F}$ is a locally constant sheaf with finite stalks then $\mathcal{F}$ is finite. It follows that in condition (4) of Definition 4.23, the words "locally constant with finite stalks" can be replaced by "finite locally constant". Also a sheaf $\mathcal{F}$ on X is constructible if and only if X can be written as a union of finitely many locally closed subschemes $Y \subset X$ for which $\mathcal{F}|_Y$ is finite locally constant (see [FK, Definitions 4.3, 4.3′], see also [Mi, Prop. 1.8]).

Definition 4.25. A complex of sheaves K is said to be **constructible** if all of its cohomology sheaves $\mathcal{H}^i(K)$ are constructible in the above sense.

4.4. **ℓ-adic sheaves.** We fix a field k, and a prime number ℓ, with ℓ distinct from the characteristic of k. Let $\bar{k}$ denote an algebraic closure of k. Let X be scheme over k of finite type, endowed with the étale topology.

Remark 4.26. A sheaf $\mathcal{F}$ in abelian groups (or in R-modules, where R is a ring) is called a **torsion sheaf** if for every section $s \in \mathcal{F}(U)$ there exists $n \in \mathbb{N}$ such that $ns = 0$. A sheaf is called an **ℓ-torsion sheaf** if all its sections are annulated by a power of ℓ. In particular, if R is an ℓ-torsion ring, then each sheaf in R-modules is an ℓ-torsion sheaf.

Let R be an **ℓ-torsion** ring (for instance $\mathbb{Z}/\ell^n\mathbb{Z}$ or a $\mathbb{Z}/\ell^n\mathbb{Z}$-algebra). The étale sheaves of R-modules over X are ℓ-torsion sheaves (see Remark 4.26).

Definition 4.27. An étale sheaf of R-modules (also called an **R-étale sheaf**) is called **finite** (resp. **with finite stalks**, resp. **locally constant**, resp. **constructible**) if it is finite (resp. with finite stalks, resp. locally constant, resp. constructible) as an étale sheaf of abelian groups.

Let $E \subset \bar{\mathbb{Q}}_\ell$ be a finite extension of $\mathbb{Q}_\ell$, with ring of integers and maximal ideal respectively denoted by $\mathfrak{o}_E$ and $\mathfrak{p}_E$.

Definition 4.28. An **$\mathfrak{o}_E$-sheaf** (also called an **ℓ-adic sheaf**, when $E = \mathbb{Q}_\ell$) is a projective system $\mathcal{F} = (\mathcal{F}_j)_{j\in\mathbb{N}}$ of étale sheaves on X such that

(1) for each j, $\mathcal{F}_j$ is a $\mathfrak{o}_E/\mathfrak{p}_E^j$-étale sheaf on X;
(2) for each j, the map $\mathcal{F}_{j+1} \mapsto \mathcal{F}_j$ is ismomorphic to the canonical map $\mathcal{F}_{j+1} \mapsto \mathcal{F}_{j+1} \otimes_{\mathfrak{o}_E} \mathfrak{o}_E/\mathfrak{p}_E^j$.

Let $\mathcal{F} = (\mathcal{F}_j)_{j\in\mathbb{N}}$ and $\mathcal{F}' = (\mathcal{F}'_j)_{j\in\mathbb{N}}$ be two $\mathfrak{o}_E$-sheaves. We set

$$\mathrm{Hom}_{\mathfrak{o}_E}(\mathcal{F}, \mathcal{F}') := \varprojlim_j \mathrm{Hom}(\mathcal{F}_j, \mathcal{F}'_j).$$

Then $\mathrm{Hom}_{\mathfrak{o}_E}(\mathcal{F}, \mathcal{F}')$ is an $\mathfrak{o}_E$-sheaf. Hence this defines an abelian category, called the **category of (étale) $\mathfrak{o}_E$-sheaves on** X.

Definition 4.29. An $\mathfrak{o}_E$-sheaf $\mathcal{F} = (\mathcal{F}_j)_{j\in\mathbb{N}}$ is called **locally constant** (resp. **constructible**) if any $\mathcal{F}_j$ is locally constant (resp. constructible) in the sense of Definition 4.23.

We shall define now the category of E**-(étale) sheaves**. Its objects are the $\mathfrak{o}_E$-sheaves, and if $\mathcal{F}$ and $\mathcal{F}'$ are two $\mathfrak{o}_E$-sheaves, we set

$$\mathrm{Hom}_E(\mathcal{F}, \mathcal{F}') := \mathrm{Hom}_{\mathfrak{o}_E}(\mathcal{F}, \mathcal{F}') \otimes_{\mathfrak{o}_E} E.$$

When viewed as a E-sheaf, $\mathcal{F}$ will be be written $\mathcal{F} \otimes_{\mathfrak{o}_E} E$. The E-sheaf $\mathcal{F} \otimes_{\mathfrak{o}_E} E$ is called **locally constant** (resp. **constructible**) if $\mathcal{F}$ is.

Definition 4.30. A $\bar{\mathbb{Q}}_\ell$**-sheaf** if any E-sheaf where E is a finite extension of $\mathbb{Q}_\ell$. A constructible locally constant $\bar{\mathbb{Q}}_\ell$-sheaf (of rank n) is called a **local system (of rank** n).

5. The category $D_c^b(X, R)$

We keep the notation of the previous section.

Definition 5.1. Let $D(X, R)$ (resp. $D^b(X, R)$) be the derived category (resp. bounded derived category) of $\mathfrak{Sh}_{\mathrm{et}}(X, R)$:

$$D(X, R) := D\left(\mathfrak{Sh}_{\mathrm{et}}(X, R)\right) \quad \text{and} \quad D^b(X, R) := D^b\left(\mathfrak{Sh}_{\mathrm{et}}(X, R)\right).$$

5.1. **The proper pull-back functor $f^!$.** Let $f\colon X \to Y$ be a morphism betweeen two k-schemes of finite type. If d is an integer such the dimension of any fibre of f is less or equal to d, the cohomological dimension of $f_!$ is less or equal to $2d$ (see for instance [FK, page 94]).

Definition 5.2. Let $\mathcal{F}$ be an étale sheaf of R-modules on X. Then $\mathcal{F}$ is called f**-soft** (or f**-mou**) if, for any étale scheme U over X, the sheaf $\mathcal{F}_U$ (see Notation 4.22) is $f_!$-acyclic (see Definition 3.13).

Because $f_!$ is of finite cohomological dimension, any étale sheaf of R-modules on X has a finite f-soft resolution.

Let $B^\bullet$ be a complex of étale sheaf of R-modules on Y, and let $I^\bullet$ be an injective resolution of L. On the other hand, let $A^\bullet$ be a finite f-soft resolution of the constant étale sheaf.

Definition 5.3. The **proper pull-back** by f of $B^\bullet$, denoted by $f^!B^\bullet$, is the complex of sheaves defined as, a complex of presheaves, by

$$(f^!B^\bullet)^j(U) := \prod_{i\in\mathbb{Z}} \mathrm{Hom}\left((f_!(A^\bullet)_U)^i, I^{i+j}\right).$$

The functor $f^!$ defined in Definition 5.3 has a left adjoint that we shall denote $Rf_!$. In general, the functor $Rf_!$ is not the derived functor of the functor $f_!$ that we defined in Definition 4.17. We shall denote that derived functor $R(f_!)$. Nevertheless, if X is a subscheme of a scheme Y and $f\colon X \to Y$ is the corresponding immersion, then $f^!$ is the right derived functor of the functor $f^!$ that we defined in Definition 4.19 and $Rf_!$ is the right derived functor of the functor $f_!$. If f is a finite morphism, we also have $Rf_! = Rf_* = R(f_!)$.

Definition 5.4. Let $D^b_c(X,R)$ denote the full subcategory of $D^b(X,R)$ consisting of constructible (see Definition 4.25) complexes of sheaves.

Remark 5.5. The category $D^b_c(X,R)$ is triangulated.

Let x be a point in X. We denote by $d(x)$ the dimension of $\overline{\{x\}}$, the closure of $\{x\}$. Let $\overline{x}$ be a geometric point with image x in X.

Definition 5.6. We set

$${}^pD^{b,\leq 0}_c(X,R) := \left\{K \text{ object of } D^b_c(X,R) \ : \ (\mathcal{H}^iK)_{\overline{x}} = 0 \text{ for all } x \in X \text{ if } i > -d(x)\right\},$$

$${}^pD^{b,\geq 0}_c(X,R) := \left\{K \text{ object of } D^b_c(X,R) \ : \ (\mathcal{H}^iK)_{\overline{x}} = 0 \text{ for all } x \in X \text{ if } i < -d(x)\right\}.$$

Remark 5.7. The condition $(\mathcal{H}^iK)_{\overline{x}} = 0$ for all $x \in X$ if $i > -d(x)$ is equivalent to the following support condition:

$$\dim(\mathrm{Supp}(\mathcal{H}^iK)) \leq -i,$$

where Supp is defined by Definition 4.10. (In particular, we have $\mathcal{H}^iK = 0$ for $i > 0$.)

On the other hand, ${}^pD^{b,\geq 0}_c(X,R)$ is the full subcategory of $D^b_c(X,R)$ whose objects are those K in $D^b_c(X,R)$ such that $\mathbb{D}K$ is an object of ${}^pD^{b,\leq 0}_c(X,R)$.

Theorem 5.8. *The pair $({}^pD^{b,\leq 0}_c(X,R), {}^pD^{b,\geq 0}_c(X,R))$ defines a t-structure (called the* **selfdual** *or* **middle perversity t-structure***) on $D^b_c(X,R)$.*

Definition 5.9. The heart of the t-structure $({}^pD^{b,\leq 0}_c(X,R), {}^pD^{b,\geq 0}_c(X,R))$ (see Definition 2.4) is called the category of **perverse sheaves** on R-modules on X.

The rest of this section will be devoted to the proof of Theorem 5.8. The idea of the proof is the following: define a filtration of $D^b_c(X,R)$ by full subcategories on which we can build a t-structure by using the gluing method explained in [BBD, § 1.4.6], then derive a t-structure on $D^b_c(X,R)$ by passing to the limit, and finally check that the obtained t-structure coïncides with $({}^pD^{b,\leq 0}_c(X,R), {}^pD^{b,\geq 0}_c(X,R))$.

5.2. The filtration of $D^b_c(X,R)$.

Definition 5.10. Let X be a k-scheme of finite type. A **smooth stratification** of X is a finite set $\mathcal{S}$ of subsets (called **strata**) of X such that

- Each stratum is a locally closed subset of X. (A set is **locally closed** if it is the intersection of an open set and a closed set.)
- For each stratum S, the scheme $(S_{\bar{k}})_{\mathrm{red}}$ is smooth over k and equidimensional.
- $\mathcal{S}$ is a partition of X, *i.e.,* as a set X is the disjoint union of all the strata.
- The closure of a stratum is a union of strata, *i.e.,* for each stratum S, $\bar{S} = \bigsqcup_{S' \in \mathcal{S}, S' \subset \bar{S}} S'$.

Let r denote the number of strata in $\mathcal{S}$.

Definition 5.11. Let $\mathcal{S}$, $\mathcal{S}'$ be two smooth stratifications of X. The stratification $\mathcal{S}'$ is **finer** than $\mathcal{S}$ if each stratum of $\mathcal{S}$ is a union of strata of $\mathcal{S}'$.

Definition 5.12. Let $\Sigma(R)$ be the set of pairs $(\mathcal{S}, \mathfrak{L})$ satisfying:

(1) $\mathcal{S}$ is a smooth stratification of X.

(2) $\mathfrak{L}$ is a map which associates to each stratum S in $\mathcal{S}$ a finite set $\mathfrak{L}(S)$ of isomorphy classes of $\mathfrak{Sh}(S,R)$ such that for each $[\mathcal{F}] \in \mathfrak{L}(S)$ the following holds:
   (a) $\mathcal{F}$ is a constructible locally constant sheaf of R-modules.
   (b) $\mathcal{F}$ is irreducible in the category of constructible locally constant sheaves on R-modules on S.

Definition 5.13. Let $(\mathcal{S},\mathfrak{L})$, $(\mathcal{S}',\mathfrak{L}')$ be two elements in $\Sigma(R)$. One says that $(\mathcal{S}',\mathfrak{L}')$ is **refinement** of $(\mathcal{S},\mathfrak{L})$ (one writes $(\mathcal{S}',\mathfrak{L}') \leq (\mathcal{S},\mathfrak{L})$) if

(1) $\mathcal{S}'$ is finer than $\mathcal{S}$;
(2) For any $(S,S') \in \mathcal{S} \times \mathcal{S}'$ such that $S' \subset S$, and any $[\mathcal{F}] \in \mathfrak{L}(S)$, we have $[\mathcal{F}|_{S'}] \in \mathfrak{L}(S')$.

Definition 5.14. Let $(\mathcal{S},\mathfrak{L}) \in \Sigma(R)$. An object $\mathcal{F}$ of $\mathfrak{Sh}(X,R)$ is called **$(\mathcal{S},\mathfrak{L})$-constructible** if for any stratum S of $\mathcal{S}$, the restriction of $\mathcal{F}$ to S is obtained by repeated extensions of sheaves whose isomorphy classes belong to $\mathfrak{L}(S)$.

Remark 5.15. Since the category of constructible sheaves is stable by extensions, any $(\mathcal{S},\mathfrak{L})$-constructible sheaf is constructible.

Lemma 5.16. *Let $(\mathcal{S},\mathfrak{L}) \in \Sigma(R)$. The full subcategory of $\mathfrak{Sh}(S,R)$ consisting of $(\mathcal{S},\mathfrak{L})$-constructible sheaves is stable by kernels, cokernels and extensions.*

Lemma 5.16 shows that the above definitions can be extended to derived categories as follows.

Definition 5.17. Let $(\mathcal{S},\mathfrak{L}) \in \Sigma(R)$.

- Let $D^b_{(\mathcal{S},\mathfrak{L})}(X,R)$ denote the full subcategory of $D^b(X,R)$ consisting of complexes with $(\mathcal{S},\mathfrak{L})$-constructible cohomology sheaves. The objects of $D^b_{(\mathcal{S},\mathfrak{L})}(X,R)$ are called **$(\mathcal{S},\mathfrak{L})$-constructible**.
- If U is a locally closed subset of X, union of strata in $\mathcal{S}$, let $D^b_{(\mathcal{S},\mathfrak{L})}(U,R)$ denote the triangulated category of complexes on U with $(\mathcal{S},\mathfrak{L})$-constructible cohomology sheaves on U.

Lemma 5.18. *Let $(\mathcal{S},\mathfrak{L}) \in \Sigma(R)$, let U and V be two locally closed subsets of X, unions of strata in $\mathcal{S}$, such that $U \subset V$, and let $j\colon U \to V$ the corresponding immersion morphism. Then the functors Rj^* and $Rj_!$ preserve the filtration of $D^b_c(X,R)$ by $\Sigma(R)$:*

- *If K is a complex in $D^b_{(\mathcal{S},\mathfrak{L})}(V,R)$, the Rj^*K is in $D^b_{(\mathcal{S},\mathfrak{L})}(U,R)$.*
- *If K is a complex in $D^b_{(\mathcal{S},\mathfrak{L})}(U,R)$, the $Rj_!K$ is in $D^b_{(\mathcal{S},\mathfrak{L})}(V,R)$.*

In general, the functors Rj_* and $Rj^!$ do not preserve the filtration of $D^b_c(X,R)$ by $\Sigma(R)$. In order to repair this, we shall consider a refinement of the filtration.

Definition 5.19. Let $\Sigma_0(R)$ be the subset of $\Sigma(R)$ consisting of the pairs $(\mathcal{S},\mathfrak{L})$ which satisfy:

$$\forall S \in \mathcal{S}, \forall \mathcal{F} \in \mathfrak{L}(S), \forall i, \;\; R^i\iota_*\mathcal{F} \text{ is } (\mathcal{S},\mathfrak{L})\text{-constructible},$$

where $\iota\colon S \to X$ is the canonical immersion.

Lemma 5.20. *Let $(\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)$, let U and V be two locally closed subsets of X, unions of strata in $\mathcal{S}$, such that $U \subset V$, and let $j\colon U \to V$ the corresponding immersion morphism. Then we have:*

- *If K is a complex in $D^b_{(\mathcal{S},\mathfrak{L})}(U,R)$, the Rj_*K is in $D^b_{(\mathcal{S},\mathfrak{L})}(V,R)$.*
- *If K is a complex in $D^b_{(\mathcal{S},\mathfrak{L})}(V,R)$, the $Rj^!K$ is in $D^b_{(\mathcal{S},\mathfrak{L})}(U,R)$.*

Lemma 5.21. *Let $(\mathcal{S},\mathfrak{L}) \in \Sigma(R)$. One can refine the stratification $\mathcal{S}$ and add a finite number of isomorphy classes of locally constant sheaves on each stratum of the refined stratification in order to get a pair $(\mathcal{S}',\mathfrak{L}')$ which belongs to $\Sigma_0(R)$.*

Lemma 5.21 shows that $\Sigma_0(R)$ provides a filtration of $D^b_c(X,R)$. We have

$$D^b_c(X,R) = \bigcup_{(\mathcal{S},\mathfrak{L})\in\Sigma_0(R)} D^b_{(\mathcal{S},\mathfrak{L})}(X,R). \tag{2}$$

5.3. **The t-structure.** We shall define on each step of the filtration a t-structure by using the gluing method.

For each $(\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)$, we set

$${}^rD^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) := \left\{K \in D^b_{(\mathcal{S},\mathfrak{L})}(X,R) \,:\, R^i(\iota_S^* K) = 0 \;\; \forall S \in \mathcal{S} \text{ and } i > -\dim S\right\}$$

$${}^rD^{b,\geq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) := \left\{K \in D^b_{(\mathcal{S},\mathfrak{L})}(X,R) \,:\, R^i(\iota_S^! K) = 0 \;\; \forall S \in \mathcal{S} \text{ and } i < -\dim S\right\}.$$

Proposition 5.22. *Let $(\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)$. Then $({}^rD^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,R), {}^rD^{b,\geq 0}_{(\mathcal{S},\mathfrak{L})}(X,R))$ is a t-structure on $D^b_{(\mathcal{S},\mathfrak{L})}(X,R)$.*

Proposition 5.23. *(Compatibility of t-structures) Let $(\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)$ and $(\mathcal{S}',\mathfrak{L}') \in \Sigma_0(R)$ such that $(\mathcal{S},\mathfrak{L}) \leq (\mathcal{S}',\mathfrak{L}')$. Then we have the following inclusions between full subcategories:*

$${}^rD^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) \subset {}^rD^{b,\leq 0}_{(\mathcal{S}',\mathfrak{L}')}(X,R) \quad \text{and} \quad {}^rD^{b,\geq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) \subset {}^rD^{b,\geq 0}_{(\mathcal{S}',\mathfrak{L}')}(X,R).$$

We set

$${}^rD^{b,\leq 0}_{c}(X,R) := \left\{K \in D^b(X,R) \,:\, K \in {}^rD^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) \text{ for some } (\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)\right\},$$

$${}^rD^{b,\geq 0}_{c}(X,R) := \left\{K \in D^b(X,R) \,:\, K \in {}^rD^{b,\geq 0}_{(\mathcal{S},\mathfrak{L})}(X,R) \text{ for some } (\mathcal{S},\mathfrak{L}) \in \Sigma_0(R)\right\}.$$

Then $({}^rD^{b,\leq 0}_{c}(X,R), {}^rD^{b,\geq 0}_{c}(X,R))$ is a t-structure on $D^b_c(X,R)$.

The next Proposition concludes the proof of Theorem 5.8

Proposition 5.24. *We have ${}^rD^{b,\leq 0}_{c}(X,R) = {}^pD^{b,\leq 0}_{c}(X,R)$ and ${}^rD^{b,\geq 0}_{c}(X,R) = {}^pD^{b,\geq 0}_{c}(X,R)$.*

6. The category $D^b_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)$

As before, let E be a finite extension of $\bar{\mathbb{Q}}_\ell$, let $\mathfrak{o}_E$ denote its ring of integers, let $\mathfrak{p}_E$ be the maximal ideal of $\mathfrak{o}_E$, and let $\kappa_E := \mathfrak{o}_E/\mathfrak{p}_E$ denote the residue field of E. We shall apply some of the constructions of the previous section with $R = \kappa_E$. We fix $(\mathcal{S},\mathfrak{L}) \in \Sigma_0(\kappa_E)$.

Definition 6.1. (*cf.* [BBD, § 2.2.13]) We fix $j \in \mathbb{N}$. An object $\mathcal{F}$ of $\mathfrak{Sh}(X, E/\mathfrak{p}_E^j)$ is called **$(\mathcal{S},\mathfrak{L})$-constructible** if, for any $m < j$, the κ_E-sheaf $\mathfrak{p}_E^m\mathcal{F}/\mathfrak{p}_E^{m+1}\mathcal{F}$ is $(\mathcal{S},\mathfrak{L})$-constructible in the sense of Definition 5.14.

Definition 6.2. (*cf.* [BBD, § 2.2.14, § 2.2.17]) An **$(\mathcal{S},\mathfrak{L})$-constructible $\mathfrak{o}_E$-complex** is a collection $K = (K_n)_{n\in\mathbb{N}}$ of objects K_n of $D^b(X, \mathfrak{o}_E/\mathfrak{p}_E^n)$ satisfying the following conditions:

(1) For each n, K_n is of finite Tor-dimension (see Definition 3.19).
(2) For any $m \leq n$, we have

$$K_n \otimes^L_{\mathfrak{o}_E/\mathfrak{p}_E^n} \mathfrak{o}_E/\mathfrak{p}_E^m = K_m.$$

(3) For any i and any n, $\mathcal{H}^i K_n$ is an $(\mathcal{S},\mathfrak{L})$-constructible $E/\mathfrak{o}_E^n$-sheaf on X in the sense of Definition 6.1.

The i**th cohomology "sheaf" of** K, denoted $\mathcal{H}^i K$, is the projective system $\mathcal{H}^i K := (\mathcal{H}^i K_n)$. Hence $\mathcal{H}^i K$ is an object of the projective limit category

$$\varprojlim_n \mathfrak{Sh}(X, \mathfrak{o}_E/\mathfrak{p}_E^n).$$

Let $K = (K_n)_{n\in\mathbb{N}}$ and $K' = (K'_n)_{n\in\mathbb{N}}$ be two $(\mathcal{S},\mathfrak{L})$-constructible $\mathfrak{o}_E$-complexes. We set

$$\mathrm{Hom}_{D^b_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)}(K,K') := \varprojlim_{n\in\mathbb{N}} \mathrm{Hom}_{D^b(X,\mathfrak{o}_E/\mathfrak{p}_E^n)}(K_n, K'_n).$$

The category we get will be denoted $D^b_c(X,\mathfrak{o}_E)$.

Remark 6.3. Let $E'' \supset E$ be a finite extension of $\mathbb{Q}_\ell$, and let $(\mathcal{S}'',\mathfrak{L}'') \in \Sigma_0(\kappa_E)$. Then we have a faithful functor $-\otimes^L_{\mathfrak{o}_E} \mathfrak{o}_{E''}$ from $D^b_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)$ to $D^b_{(\mathcal{S}'',\mathfrak{L}'')}(X,\mathfrak{o}_{E''})$.

We set

$$D^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E) := \left\{ K \in D^b_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E) \,:\, \mathcal{H}^i K = o \ \ \forall i > 0 \right\}.$$

The embedding functor from $D^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)$ into $D^b_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)$ has a right adjoint, but it is not as easy to construct as for derived categories (it has been constructed by Deligne in [Del, p. 149]).

Proposition 6.4. *If $\mathfrak{o}_E$ is contained in $\mathfrak{o}_{E'}$, and if $(\mathcal{S}',\mathfrak{L}')$ is refinement of $(\mathcal{S},\mathfrak{L})$, then there is a canonical embedding functor from $D^{b,\leq 0}_{(\mathcal{S},\mathfrak{L})}(X,\mathfrak{o}_E)$ into $D^{b,\leq 0}_{(\mathcal{S}',\mathfrak{L}')}(X,\mathfrak{o}_{E'})$, and this functor is t-exact (in the sense of Definition 2.13).*

7. The category $D^b_c(X,\bar{\mathbb{Q}}_\ell)$

7.1. **Definition.**

Definition 7.1. A **constructible** $\bar{\mathbb{Q}}_\ell$**-complex** on X is a triple $\underline{K} := (E,(\mathcal{S},\mathfrak{L}),K)$ where

(1) E is a finite extension of $\bar{\mathbb{Q}}_\ell$;
(2) $(\mathcal{S},\mathfrak{L})$ is an element of $\Sigma_0(\kappa_E)$;
(3) K is an $(\mathcal{S},\mathfrak{L})$-constructible $\mathfrak{o}_E$-complex.

Definition 7.2. Let $D^b_c(X,\bar{\mathbb{Q}}_\ell)$ be the category of $\bar{\mathbb{Q}}_\ell$-constructible complexes, with morphisms defined as follows: If $(E,(\mathcal{S},\mathfrak{L}),K)$ and $(E',(\mathcal{S}',\mathfrak{L}'),K')$ are two such complexes, let E'' be a finite extension $\bar{\mathbb{Q}}_\ell$ which contains as subextensions both E and E', let $(\mathcal{S}'',\mathfrak{L}'')$ be an element of $\Sigma_0(\kappa_{E''})$ which a refinement of both $(\mathcal{S},\mathfrak{L})$ and $(\mathcal{S}',\mathfrak{L}')$, and set

$$\begin{aligned} K \otimes^L_{\mathfrak{o}_E} \mathfrak{o}_{E''} &:= \left(K_n \otimes^L_{\mathfrak{o}_E/\mathfrak{p}_E^n} \mathfrak{o}_{E''}/\mathfrak{p}_{E''}^n\right)_{n\in\mathbb{N}} \\ K' \otimes^L_{\mathfrak{o}_{E'}} \mathfrak{o}_{E''} &:= \left(K'_n \otimes^L_{\mathfrak{o}_{E'}/\mathfrak{p}_{E'}^n} \mathfrak{o}_{E''}/\mathfrak{p}_{E''}^n\right)_{n\in\mathbb{N}}. \end{aligned} \tag{3}$$

All the constructions which have been made on $D(x, \mathfrak{o}_E/\mathfrak{p}_E^j)$ can be extended to the category $D_c^b(X, \bar{\mathbb{Q}}_\ell)$, which is then equipped with a triangulated category structure on which the six functors Rf_*, Rf^*, $Rf_!$, $f^!$, $R\,\mathrm{Hom}$, $-\otimes^L -$, and the functors $\mathcal{H}^i$ can be defined and keep their properties (adjunction, etc.).

7.2. **The Grothendieck-Verdier duality.** Let $c_X \colon X \to \mathrm{Spec}(k)$ be the canonical projection.

Definition 7.3. The **dualizing complex** of X is the complex

$$\mathbb{D}_X := c_X^! \bar{\mathbb{Q}}_\ell,$$

defined as the proper pull-back image by c_X of the constant sheaf $\bar{\mathbb{Q}}_\ell$ on $\mathrm{Spec}(k)$

.

Definition 7.4. For any constructible $\bar{\mathbb{Q}}_\ell$-complex $\underline{K}$ on X, we define its **dual** by

$$\mathrm{D}_X(\underline{K}) := R\,\mathrm{Hom}(\underline{K}, \mathbb{D}_X).$$

The functor D_X is called the **Grothendieck-Verdier duality**.

Theorem 7.5. *The functor* D_X *is a self-equivalence category of* $D(X)$*, and that we have*

$$\mathrm{D}_X^2 \simeq \mathrm{Id}_{D(X)}.$$

It follows that

$$\mathrm{D}_X(Rf_*) = Rf_! \quad \text{and} \quad \mathrm{D}_X(Rf^*) = f^!.$$

Notation 7.6. In order to simplify the notation, we shall write

$$D(X) := D_c^b(X, \bar{\mathbb{Q}}_\ell).$$

7.3. **The category of perverse ($\bar{\mathbb{Q}}_\ell$)-sheaves.**

7.3.1. *Definition of the category.* Proposition 6.4 allows us to define a t-structure on $D(X)$, denoted $({}^pD(X)^{\leq 0}, {}^pD(X)^{\geq 0})$, as the inductive limit of the perversity t-structures on the $D^b_{(\mathcal{S},\mathfrak{L})}(X, \mathfrak{o}_E)$. This t-structure will be called the **perversity t-structure**.

Notation 7.7. From now on, any object which refers to the perversity t-structure will be denoted with a left exponent p.

The Grothendieck-Verdier duality sends ${}^pD(X)^{\leq 0}$ to ${}^pD(X)^{\geq 0}$.

Definition 7.8. The heart of the t-structure $({}^pD(X)^{\leq 0}, {}^pD(X)^{\geq 0})$

$$\mathcal{M}(X) := {}^pD(X)^{\leq 0} \cap {}^pD(X)^{\geq 0},$$

is called the *category of perverse ($\bar{\mathbb{Q}}_\ell$)-sheaves.*

We can summerize the above description as follows.

- ${}^pD(X)^{\leq 0}$ is the full subcategory of $D(X)$ whose objects are those K in $D(X)$ such that, for any integer i, ${}^p\mathcal{H}^i K := {}^t\mathcal{H}^i K$ (see Definition 2.10) has support of dimension $\leq -i$.
- ${}^pD(X)^{\leq 0}$ is the full subcategory of $D(X)$ whose objects are those K in $D(X)$ such that $\mathrm{D}(K) \in {}^pD(X)^{\leq 0}$.

Proposition 7.9. [BBD, 2.14, 1.3.6, 4.3.1] *The category* $\mathcal{M}(X)$ *is an abelian category in which all objects have finite length.*

Definition 7.10. An object K of $D(X)$ is said to be **semisimple** if K is isomorphic to the direct sum

$$\bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i K[-i],$$

and if ${}^p\mathcal{H}^i(K)$ is semisimple for any $i \in \mathbb{Z}$.

7.3.2. *Middle extension functor.*

Proposition 7.11. [BBD, p. 68] *If $f\colon X \to Y$ is a quasifinite (i.e., with finite fibres) morphism, then the functors $Rf_!$ and Rf^* between $D(X)$ and $D(Y)$ are right t-exact (see Definition 2.13). (By adjunction, the functors Rf_* and $f^!$ are left t-exact.)*

Proposition 7.12. [BBD, p. 102] *If $f\colon X \to Y$ is an affine morphism, then the functors $Rf_*\colon D(X) \to D(Y)$ is right t-exact.*

Let $f\colon X \to Y$ be a quasifinite morphism. Using Proposition 7.11, we define the following four functors between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$:

$${}^pf_* := {}^p\tau_{\leq 0}Rf_*, \quad {}^pf^* := {}^p\tau_{\leq 0}Rf^*, \quad {}^pf_! := {}^p\tau_{\geq 0}Rf_!, \quad {}^pf^! := {}^p\tau_{\geq 0}f^!.$$

Let K be an object of $\mathcal{M}(X)$. We always have a natural morphism

$$Rf_!K \longrightarrow Rf_*K. \tag{4}$$

Because Rf_*K is an object of ${}^pD(X)^{\geq 0}$ (this follows from Prop. 7.11), the morphism (4) factors through ${}^pf_!K$. Now, since ${}^pf_!K$ is an object of ${}^pD(X)^{\leq 0}$, the morphism

$${}^pf_!K \longrightarrow Rf_*K \tag{5}$$

itself factors through ${}^pf_*K$. Hence we can define a perverse sheaf on Y by setting:

$$f_{!*}K := \operatorname{im}({}^pf_!K \to {}^pf_*K). \tag{6}$$

Definition 7.13. The functor $f_{*!}\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ is called the **middle extension functor**.

7.3.3. *Simple objects in the category of perverse sheaves.* Let S be a locally closed smooth irreducible subscheme of X, and let $\mathcal{E}$ be a local system on S. Let $\iota_S\colon S \hookrightarrow X$ denote the inclusion of S into X, and let $\iota_{\overline{S}}\colon \overline{S} \hookrightarrow X$ be the inclusion of the closure of S into X. Let $\mathrm{IC}(\overline{S}, \mathcal{E})i \in D(\overline{S})$ be the corresponding **intersection cohomology complex** defined by Goresky-Mac Pherson and Deligne. The complex

$$K := \mathrm{IC}(\overline{S}, \mathcal{E})[\dim S], \tag{7}$$

is characterized by the following properties

1. $\mathcal{H}^iK = 0$ if $i < -\dim S$;
2. $\mathcal{H}^{\dim S}K|_S = \mathcal{E}$;
3. $\dim(\mathrm{Supp}(\mathcal{H}^iK)) < -i$ if $i > -\dim S$,
4. $\dim \mathrm{Supp}(\mathcal{H}^i\mathrm{D}_XK)) < -i$ if $i > -\dim S$.

Hence K is a perverse sheaf on $\overline{S}$. The functor

$$(\iota_S)_{!*}\colon \mathcal{M}(S) \longrightarrow \mathcal{M}(X)$$

is fully faithful and satisfies

$$(\iota_S)_{!*}(\mathcal{E}[\dim S]) = (\iota_{\overline{S}})_!\left(\mathrm{IC}(\overline{S}, \mathcal{E})[\dim S]\right). \tag{8}$$

Definition 7.14. We say that (8) is the **perverse extension of $\mathcal{E}$ to** X.

Theorem 7.15. [BBD, Theorem 4.3.1] *The following holds:*

(1) *If $\mathcal{E}$ be an irreducible local system on S, then* $\mathrm{IC}(\overline{S}, \mathcal{E})[\dim S]$ *is a simple perverse sheaf on X (i.e., is a simple object in $\mathcal{M}(X)$).*
(2) *All simple objects in $\mathcal{M}(X)$ are obtained in this way for some pair $(S, \mathcal{E})$ as above.*

Part 2. Character sheaves

Let k be an algebraically closed field and let G be a connected reductive algebraic group over k. If G acts on an algebraic variety X and K is an object of $\mathcal{M}(X)$, we say that K is G-equivariant if $\pi^*K[\dim G] \simeq \alpha^*K[\dim G]$, where $\alpha\colon G \times X \to X$ is the map which gives the action of G on X and π is the projection of $G \times X$ on X.

Character sheaves on G are certain G-equivariant (for the conjugation action of G on itself) perverse sheaves in the category $D(G) := D_c^b(G, \bar{\mathbb{Q}}_\ell)$, which were introduced by Lusztig in [Lu3] and which provide an extremely useful geometric approach to the character theory of finite groups of Lie type (see in particular [Lu4], [Lu7], [Sh], [Sh2], [Sh3], [Wa]). For a survey on character sheaves, see [Lu5], and also [Lau]. For a course on character sheaves, see [MS]. With k an algebraic closure of a p-adic field, Cunningham and Salmasian (see [CS]) have adapted the Lusztig geometric approach to the study of the character theory of reductive p-adic groups.

Below we shall state the definition of a character sheaf following [Lu3, Definition 2.10] (with the characterization of [Lu3, Prop. 2.9.(a)]) and give some classification results. Some equivalent definitions and different points of view can be found for instance in [Gin], [MV], [Gal], [Mir]. The reader will find generalization of character sheaves in [Lu8] and [Lu9], see also [HL].

From now on, we shall assume that k is the algebraic closure of a finite field $\mathbb{F}_q$. But note that this assumption would be necessary everywhere.

8. Definition of character sheaves

We fix a Borel subgroup B of G with unipotent radical denoted U and a maximal torus $T \subset B$. Let $W := \mathrm{N}_G(T)/T$ be the Weyl group. For $w \in W$, let $\dot{w}$ be a representative of w in $\mathrm{N}_G(T)$. An element w of W may be regarded as an automorphism $w\colon T \to T$ by setting $w(t) := \dot{w}t\dot{w}^{-1}$ $(t \in T)$.

Definition 8.1. A local system $\mathcal{L}$ on T of rank 1 is called a **Kummer local system** (or a **tame local system**) if $\mathcal{L}^{\otimes n} \cong \bar{\mathbb{Q}}_\ell$ for some positive integer n, invertible in k. Let $\mathcal{S}(T)$ denote the set of isomorphism classes of Kummer local systems on T.

The group W acts on $\mathcal{S}(T)$ by $w\colon \mathcal{L} \mapsto (w^{-1})^*\mathcal{L}$, where $(w^{-1})^*$ denotes the inverse image under $w^{-1}\colon T \to T$. Let $\mathcal{L} \in \mathcal{S}(T)$. We set

$$W_{\mathcal{L}} := \left\{w \in W \ : \ (w^{-1})^*\mathcal{L} = \mathcal{L}\right\}. \tag{9}$$

(This group is denoted $W'_{\mathcal{L}}$ in [Lu3, § (2.2.1)].)

If $g, h \in G$, and K is a subgroup of G, we shall set ${}^gK := gKg^{-1}$ and $g^h := h^{-1}gh$. Let $\mathcal{B}$ denote the variety of all Borel subgroups of G. For each $w \in W$, let $O(w)$ be the subvariety of $\mathcal{B} \times \mathcal{B}$ defined by

$$O(w) := \left\{(B', B'') \in \mathcal{B} \times \mathcal{B} \ : \ \text{ for some } g \in G \text{ such that } {}^gB' = B \text{ and } {}^gB'' = {}^{\dot{w}}B\right\},$$

and let

$$Y_w := \{(g, B') \in G \times \mathcal{B} \;:\; (B', {}^gB') \in O(w)\}\,.$$

Let $\pi_w \colon Y_w \to G$ be the morphism which is defined by the first projection, that is, $\pi_w(g, B') := g$.

Let $\mathrm{pr}_{\dot w} \colon BwB \to T$ the map which is defined by

$$\mathrm{pr}_{\dot w}(u\dot w t u') := t, \quad \text{for } u, u' \in U \text{ and } t \in T,$$

and and let

$$\dot Y_w := \left\{(g, hU) \in G \times (G/U) \;:\; g^h \in BwB\right\}.$$

The map $(g, hU) \mapsto \mathrm{pr}_{\dot w}(g^h)$ from $\dot Y_w$ to T is T-equivariant with respect to both the action $t_0 \colon (g, hU) \mapsto (g, ht_0^{-1}(U))$ of T on $\dot Y_w$ and the action $t_0 \colon t \mapsto (t_0^{\dot w} t t_0^{-1})$ of T on itself.

Hence, if $\mathcal{L} \in \mathcal{S}(T)$ and $w \in W_{\mathcal{L}}$, the the inverse image $\dot{\mathcal{L}}$ of $\mathcal{L}$ under $\dot Y_w \to T$ is T-equivariant. The map $\dot Y_w \to Y_w$ defined by $(g, hU) \mapsto (g, {}^hB)$ is a principal fibration with group T (for the above action of T on $\dot Y_w$). It follows that there exists a unique $\bar{\mathbb{Q}}_\ell$-local system of rank 1, $\tilde{\mathcal{L}}$ on Y_w whose inverse image under $\dot Y_w \to Y_w$ is $\dot{\mathcal{L}}$. The isomorphism class of $\tilde{\mathcal{L}}$ does not depend on the choice of the representative $\dot w$ of w. We shall set for $w \in W_{\mathcal{L}}$:

$$K_w^{\mathcal{L}} := (\pi_w)_! \tilde{\mathcal{L}} \in D(G). \tag{10}$$

Definition 8.2. A **character sheaf** on G is an irreducible perverse sheaf on G which is a constituent of ${}^p\mathcal{H}^i(K_w^{\mathcal{L}})$, for some $\mathcal{L} \in \mathcal{S}(T)$, some $w \in W_{\mathcal{L}}$ and some $i \in \mathbb{Z}$. We shall denote by $\hat G$ the set of isomorphism classes of character sheaves on G.

Remark 8.3. The character sheaves on the torus T are the perverse sheaves $\mathcal{L}[d]$ ($\mathcal{L} \in \mathfrak{L}(T)$), where $d = \dim T$.

Notation 8.4. For any $\mathcal{L} \in \mathcal{S}(T)$, let $\hat G_{\mathcal{L}}$ denote the set of isomorphism classes of character sheaves which are constituent of ${}^p\mathcal{H}^i(K_w^{\mathcal{L}})$ for some $w \in W_{\mathcal{L}}$ and some $i \in \mathbb{Z}$.

Then $\hat G_{\mathcal{L}}$ is a finite set which depends only on the W-orbit of $\mathcal{L}$ in $\mathcal{S}(T)$ (see [Lu3, 2.10]). The sets $\hat G_{\mathcal{L}}$ form a partition of $\hat G$ (see [Lu3, 11.2]).

9. Cuspidal character sheaves

Let P be a parabolic subgroup of G such that $P \supset B$. Let U_P denote the unipotent radical of P and let L be the Levi subgroup of P containing T. We denote by π_P the canonical homomorphism of P onto L and by $\iota_P \colon P \hookrightarrow G$ the inclusion.

Definition 9.1. [Lu3, (3.8.1)] Let $\mathrm{res}^G_{L\subset P} \colon D(G) \to D(P)$ be the functor defined by

$$\mathrm{res}^G_{L\subset P}(A) := (\pi_P)_! \iota_P^* A\,(\dim U_P).$$

Proposition 9.2. [Lu3, Prop. 3.9] *If $A \in \hat G$, then $\mathrm{res}^G_{L\subset P} A \in D(L)$ is semisimple (see Definition 7.10). More precisely, it is a direct sum of finitely many complexes of the form $A'[n]$, where $A' \in \hat L$ and n is an integer.*

Definition 9.3. [Lu3, Def. 3.10] A character sheaf A on G is said to be **cuspidal** if for any proper parabolic subgroup P of G containing B (with Levi subgroup $L \supset T$), we have

$$\operatorname{res}^G_{L\subset P} A[-1] \in D(L)^{\leq 0},$$

or, equivalently,

$$\dim \operatorname{Supp} {}^p\mathcal{H}^i(\operatorname{res}^G_{L\subset P}(A)) < -i, \quad \text{for all } i.$$

Proposition 9.4. [Lu3, (7.1.6)] *A character sheaf A is cuspidal if and only if, for any proper parabolic subgroup P of G with Levi subgroup L, we have $\operatorname{res}^G_{L\subset P} A = 0$ in $D(L)$.*

For each $g \in G$, let g_{s} denote the semisimple part of g. Let $H_G(g)$ be the centralizer of the connected of $\mathrm{Z}^\circ_G(g_{\mathrm{s}})$. Then $H_G(g)$ is the smallest Levi subgroup of a parabolic subgroup of G which contains $\mathrm{Z}^\circ_G(g_{\mathrm{s}})$.

Definition 9.5. An element $g \in G$ (or its conjugacy class) is said to be **isolated** if $H_G(g) = G$.

Example 9.6. Take $G = \mathrm{Sp}_4(k)$ and let g be the following semisimple element:

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then g is isolated. Indeed, its centralizer is isomorphic to $\mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$.

Definition 9.7. [Lu2, Def. 2.4] Let $\mathcal{E}$ be a $G \times \mathrm{Z}^\circ_G$-equivariant local system on Σ. The pair $(\Sigma, \mathcal{E})$ is said to be **cuspidal** if

Proposition 9.8. [Lu3, (3.12)] *Let A be a cuspidal character sheaf. Then there is a unique $G \times \mathrm{Z}^\circ_G$-orbit $\Sigma \subset G$ and a unique irreducible $G \times \mathrm{Z}^\circ_G$-equivariant local system on Σ such that*

$$A = \mathrm{IC}(\overline{\Sigma}, \mathcal{E})\,[\dim \Sigma],$$

extended to the whole of G, by 0 on $G - \overline{\Sigma}$ (i.e., A is the perverse extension of $\mathcal{E}$ to G in the terminology of Definition 7.14). Moreover, the image of Σ in G/Z°_G is an isolated conjugacy class of G/Z°_G, and for each $g \in \Sigma$, the group $\mathrm{Z}^\circ_G(g)/\mathrm{Z}^\circ_G$ is unipotent.

10. Parabolic induction

Consider the diagram

$$L \xleftarrow{\pi} V_1 \xrightarrow{\pi'} V_2 \xrightarrow{\pi''} G,$$

where

$$V_1 := \left\{(g,h) \in G \times G \,:\, h^{-1}gh \in P\right\},$$
$$V_1 := \left\{(g,h) \in G \times (G/P) \,:\, h^{-1}gh \in P\right\},$$
$$\pi(g,h) := \pi_P(h^{-1}gh), \quad \pi'(g,h) := (g, hP), \quad \pi''(g,hP) := g.$$

Let $K_L \in \mathcal{M}(L)$ be an L-equivariant (for the conjugation of L on L) perverse sheaf. In [Lu3, § 4.1], Lusztig associated to K_L a complex $\operatorname{ind}^G_{L\subset P} K_L$ in $D(G)$ as follows. The perverse sheaf $\tilde{\pi} K_L \in \mathcal{M}(V_1)$ (see Notations 4.22 (2)) is P-equivariant for the action $p\colon (g,h) \mapsto (g, hp^{-1})$ of P on V_1 and for the action $p\colon l \mapsto \pi_P(p) l \pi_P(p)^{-1}$

of P on L. Since π' is a locally trivial principal P-bundle, there is a well-defined perverse sheaf $K' \in \mathcal{M}(V_2)$ such that

$$\tilde{\pi} K_L = \tilde{\pi}' K'.$$

The we set

$$\mathrm{ind}_{L \subset P}^G(K_L) := (\pi'')_! K'.$$

Proposition 10.1. *For any integer i, ${}^p\mathcal{H}^i(\mathrm{ind}_{L\subset P}^G(K_L))$ is a G-equivariant perverse sheaf on G (for the conjugation action).*

The functor $\mathrm{ind}_{L\subset P}^G \colon \mathcal{M}(L) \to D(G)$ is called the **parabolic induction functor**. It satisfies the usual transitivity property (see [Lu3, Prop. 4.2]).

Proposition 10.2. [Lu3, (4.3.2)] *If $A_L \in \hat{L}$ is cuspidal, then $\mathrm{ind}_P^G(A) \in \mathcal{M}(G)$ and is semisimple (in the sense of definition 7.10).*

Theorem 10.3. [Lu3, Theorem 4.4] *For any $A \in \hat{G}$, there exists a Levi subgroup $L \supset T$ of a parabolic subgroup $P \supset B$ of G and a cuspidal character sheaf $A_L \in \hat{L}$ such that A is a direct summand of $\mathrm{ind}_{L\subset P}^G(A_L)$.*

Moreover, the following holds:

(1) *For any $A_L \in \hat{L}$, we have $\mathrm{ind}_{L\subset P}^G(A_L) \in \mathcal{M}(G)$.*
(2) *For any $A \in \hat{G}$, we have $\mathrm{res}_{L\subset P}^G(A) \in D(L)^{\leq 0}$.*
(3) *(Frobenius reciprocity)*

$$\mathrm{Hom}_{D(L)}\left(\mathrm{res}_{L\subset P}^G(A), A_L\right) \simeq \mathrm{Hom}_{D(G)}\left(A, \mathrm{ind}_{L\subset P}^G(A_L)\right).$$

11. Unipotent support of a character sheaf

Let $\mathcal{L} \in \mathcal{S}(T)$ a fixed local system on T. In [Lu3, Corollary 16.7], Lusztig has defined a canonical surjection from $\hat{G}_{\mathcal{L}}$ (see Notation 8.4) to the set of two-sided cells of the group $W_{\mathcal{L}}$.

Let $\mathbf{c}$ be two-sided cell of $W_{\mathcal{L}}$, and let $\hat{G}_{\mathcal{L},\mathbf{c}}$ denote the subset of character sheaves on G which belongs to the fibre above $\mathbf{c}$ of this surjection. This gives a partition of the set $\hat{G}_{\mathcal{L}}$ by the set of two-sided cells of $W_{\mathcal{L}}$:

$$\hat{G}_{\mathcal{L}} = \bigsqcup_{\mathbf{c}} \hat{G}_{\mathcal{L},\mathbf{c}}.$$

For simplicity we shall assume from now on that the centre Z_G of G is connected (for the general case, see [Lu6, § 10.4–10.5]). The group $W_{\mathcal{L}}$ is then a Weyl group. To each irreducible representation E of $W_{\mathcal{L}}$ is attached a unique two-sided cell $\mathbf{c}$ of $W_{\mathcal{L}}$ (see [Lu6, § 10.4]). We shall say that E **belongs to c**.

Let $\mathbf{c}$ be a fixed two-sided cell of $W_{\mathcal{L}}$. Among the irreducible representations of $W_{\mathcal{L}}$ which belong to $\mathbf{c}$, there is a unique special representation, say $E(\mathbf{c})$.

Following Lusztig [Lu3, § 10.6], we shall say that a unipotent class $\mathcal{O}$ in G has property $(\star)$ with respect to $(\mathcal{L}, \mathbf{c})$ if

(a) there exists a character sheaf $A \in \hat{G}_{\mathcal{L},\mathbf{c}}$ and an element $g \in G$ with Jordan decomposition $g = g_{\mathrm{u}} g_{\mathrm{s}}$ (with $g_{\mathrm{u}} \in \mathcal{O}$ and g_{s} semisimple) such that the restriction of A to the G-conjugacy class of g is non-zero, and

(b) for any character sheaf $A \in \hat{G}_{\mathcal{L},\mathbf{c}}$, any unipotent class $\mathcal{O}'$ such that

$$\dim \mathcal{O}' > \dim \mathcal{O}$$

and any $g' \in G$ with unipotent part in $\mathcal{O}'$, the restriction of A to the G-conjugacy class of g' is zero.

Theorem 11.1. [Lu6, Theorem 10.7] *Assume that the characteristic p of k is large enough. Let $(\mathcal{L}, \mathbf{c})$ as above. Let $\mathcal{O}_{\mathcal{L},\mathbf{c}}$ denote the unipotent class in G which is attached to $E(\mathbf{c})$ by the Springer correspondence. Then the class $\mathcal{O}_{\mathcal{L},\mathbf{c}}$ has property $(\star)$ with respect to $(\mathcal{L}, \mathbf{c})$, and it is the unique unipotent class which has that property.*

In the case of character sheaves which do not vanish on the unipotent variety of G, we have the following refined result.

Theorem 11.2. [AA, Theorems 7.3, 7.5, Rem. 7.8] *Assume that the characteristic p of k is good for G. Let A be a character sheaf on G which does not vanish on the unipotent variety of G. Then there exists a unipotent class $\mathcal{O}_A$ in G such that*

- *the restriction of A to $\mathcal{O}_A$ is nonzero,*
- *any unipotent class $\mathcal{O}$ on which the restriction of A is nonzero is contained in the Zariski closure of $\mathcal{O}_A$,*
- *the class $\mathcal{O}_A$ is contained in the Zariski closure of $\mathcal{O}_{\mathcal{L},\mathbf{c}}$ for each $A \in \hat{G}_{\mathcal{L},\mathbf{c}}$. If A is cuspidal, then $\mathcal{O}_A = \mathcal{O}_{\mathcal{L},\mathbf{c}}$.*

Remark 11.3. As observed in [AA, Rem. 7.7], in general (this occurs for instance when G is of exceptional type F_4) the class $\mathcal{O}_A$ can be distinct from the class $\mathcal{O}_{\mathcal{L},\mathbf{c}}$.

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